Competition and Sequential Research and Development

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Abstract

This paper advances our understanding of the interplay between R&D and competition by modeling R&D as a sequential information acquisition process under varying competitive intensities. Using a continuous-time Bayesian learning framework, it demonstrates that while higher competition generally spurs R&D investment (measured by time dedicated to it), the probability of successful innovation implementation hinges on the competitive impact on the innovation's incremental value, its change with the prior, and the replacement effect. The paper also shows that when firms can pursue correlated R&D ideas sequentially, incentives to invest in an initial idea increase with the idea's correlation when the learning speed of the initial idea is higher relative to subsequent ideas.

Keywords: Bayesian Learning, Optimal Stopping, R&D, Competition Intensity, Incremental Value of Innovation, Replacement Effect.

JEL-Classification: G32, J24, L26, M13

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1 Introduction

What is the optimal innovation strategy when firms must learn about the value of their ideas over time, and when market competition dynamically shapes innovation incentives? This paper develops a continuous-time model of sequential R&D in which firms acquire information over time about uncertain innovations, and where market structure affects both experimentation behavior and implementation decisions. Our goal is to understand how competition influences R&D outcomes in settings where innovation is costly, learning is gradual, and technologies are not mutually exclusive.

The relationship between competition and innovation has long been a topic of debate. Schumpeter (1942) argued that innovation requires the temporary monopolies and abnormal profits that arise when firms pioneer new products and processes. In contrast, Arrow (1962) famously showed that monopolists may underinvest in innovation because successful innovation cannibalizes their own rents—a phenomenon known as the *replacement effect*. Subsequent work has introduced additional mechanisms. For example, Gilbert and Newbery (1982) highlight the *preemptive effect*, whereby incumbents innovate to forestall competitive entry, while Reinganum (1983) shows that this effect is strongest under non-drastic innovation.

Although these theories are often seen as opposing, more recent research emphasizes that the relationship is context-dependent: innovation may be encouraged or discouraged by competition depending on the intensity of rivalry, the nature of technology, and the structure of the innovation process (see, e.g., Aghion et al. (2001), Aghion et al. (2005), Shapiro (2012)). Most existing models, however, focus on winner-take-all races or assume innovation is a one-shot game. This paper departs from these frameworks by modeling R&D as a sequential learning process under uncertain innovation outcomes.

Our model conceptualizes R&D as a process of gradual information acquisition à la Wald (1945), where firms observe signals over time and update their beliefs according to Bayes' rule. A Brownian signal process drives learning, and firms face a trade-off between delaying implementation to acquire more information and incurring ongoing R&D costs. Importantly, we consider a setting where ideas are not mutually exclusive: multiple firms can implement similar innovations without fully displacing rivals, and ideas can be correlated and investigated sequentially.

The main contributions of the paper are as follows. First, we characterize the optimal R&D strategy of a firm deciding whether to implement, reject, or continue learning about an uncertain innovation. Second, we extend the model to accommodate sequential ideas with correlated payoffs and demonstrate how correlation influences experimentation and implementation decisions. Third, we introduce strategic interaction between two firms, allowing one firm (the follower) to condition its behavior on the other's (the leader) decision. We demonstrate how competition can induce either preemptive or encouraging effects, depending on its impact on the incremental value of innovation. These results yield new insights into how market structure influences innovation dynamics in more general settings than those previously studied.

The rest of the paper is organized as follows: The next section presents real-world applications that can be understood through the model. Section 2 reviews the related literature. Section 3 presents the model and discusses real-world situations that align with the model's main features. Section 4 studies the solo firm's R&D decisions with one and two ideas. Section 5 analyzes the strategic interaction between a leader and a follower. Section 6 presents the main conclusions.

2 Related Literature

This paper builds on and contributes to several strands of literature at the intersection of R&D, sequential learning, and competition.

Sequential learning and experimentation. We model R&D as an optimal stopping problem with sequential information acquisition in the spirit of Wald (1945); Wald and Wolfowitz (1948). In contrast to classical sequential analysis, which often assumes learning occurs at discrete time intervals (e.g., Chernoff, 1959, 1972; Siegmund, 1985), we adopt a continuous-time Bayesian framework in which beliefs evolve as a martingale diffusion process. The agent observes a noisy signal over time, modeled as a Brownian motion with unknown drift, and updates beliefs accordingly.

This approach is closely related to the drift-diffusion model (DDM) literature, which has been extensively applied in economics and decision theory (e.g., Roberts and Weitzman, 1981; Ulu and Smith, 2009; Branco et al., 2012; Fudenberg et al., 2018; Lang, 2019; Balmaceda et al., 2025). Our framework is particularly close to Moscarini and Smith (2001), who model experimentation as a stopping problem where the firm endogenously chooses the precision of the signal. In contrast, our model assumes exogenous learning: the firm cannot influence the speed or quality of information. Instead, we focus on how competition shapes the firm's experimentation and implementation decisions when learning is costly and outcomes are uncertain.

Competition and innovation. The relationship between competition and R&D is classically ambiguous. In static reduced-form models, Boone (2000, 2001) provide conditions under which increased competition can either promote or discourage innovation, depending on how competition affects the marginal return to investment. Schmutzler (2013) generalizes these results to asymmetric duopolies, showing that the direction of the effect hinges on whether firms' investments are strategic complements or substitutes, and how competition influences the marginal gains from innovation.

Other static models (e.g., Vives, 2008; López and Vives, 2016; Letina, 2016) show that increased competition can increase the diversity of innovation strategies and reduce duplication. These models highlight that the overall R&D effort in equilibrium can rise or fall with competition intensity, depending on how profits and innovation externalities are structured.

Dynamic models: replacement and preemption. The dynamic literature has focused primarily on patent races and the tension between Arrow's *replacement effect* (Arrow, 1962) and the *preemptive motive* first identified by Gilbert and Newbery (1982). These models often assume memoryless innovation processes (e.g., exponential arrival times), with constant R&D intensity over time (see Reinganum, 1982, 1983; Lee and Wilde, 1980), which are optimal, as shown by Malueg and Tsutsui (1997). In such settings, innovation is typically modeled as a stochastic resource allocation problem, where firms compete for a fixed innovation prize whose value does not depend on the information accumulated during the R&D process. In these R&D races, being a leading or a lagging firm is immaterial since the equilibrium strategies are independent of firms' knowledge stocks and thereby they invest the same in R&D. Hence, in memoryless models, there is no sense in which one can properly speak of one competitor being ahead of another, or of the two competitors being neck-and-neck.

Recent refinements include Parra (2019), who studies a memoryless model. He shows that strong forward protection increases expected license fees, internalizing the profit-loss of the leader, discouraging followers from investing in R&D; Etro (2004), who emphasizes pre-commitment in innovation strategies; and Marshall and Parra (2019) characterize how competition and profit gaps interact to determine industry-level innovation. Gilbert et al. (2018) study Aghion et al.'s (2005) model with more than two firms and provide conditions under which the relationship between competition and innovation takes an inverted-U shape.

Our contribution. This paper contributes to the literature by modeling R&D as a dynamic process of sequential information acquisition with uncertain outcomes and costly experimentation. Unlike classical patent race models, we do not assume memoryless innovation or winner-take-all outcomes. Instead, we allow innovations to be non-drastic and non-exclusive, such that the implementation of one firm's innovation does not fully eliminate the value of a competitor's innovation. This setup is particularly relevant for industries where patents offer limited protection and technologies evolve incrementally.

3 The Model

3.1 Setting

Let's consider an industry with n firms, indexed by $i \in \mathcal{I} \equiv \{1, \ldots, n\}$, with the same instantaneous discount factor r that engage in an unmodeled product market competition. One of the firms, let's say firm i, has an idea that can be either "bad" or "good". The idea's type, denoted by θ, is unknown, with $\theta = B$ when it is bad and $\theta = G$ when it is good. The prior belief -common to every firm- that the innovation is bad is denoted by $\delta^i = \mathbb{P}(\theta^i = B) \in (0, 1)$.

At $t = 0$, firm i has three options: (i) implement the idea immediately $d^i = 1$, (ii) discard it immediately $d^i = 0$ and keep producing with the current technology, or (iii) spend time doing R&D to gather information about the profitability of the idea before deciding whether to implement it or to discard it. The firm's decision to implement it or not is irrevocable. While conducting R&D, the firm produces using the current technology and pays the instantaneous flow cost of c.

Firm i's profits when the idea is not implemented, called regular profits, are $\pi^i(0; \mu)$, where μ is a parameter profile measuring competition intensity. When the firm implements its idea, firm i's profits are $\pi^{iG}(1; \mu)$ when the idea is good and $\pi^{iB}(1; \mu)$ when it is bad.

The profit function must be understood as the equilibrium profits of a sub-game that occurs after firms observe (d, μ), where firms compete in the product market by simultaneously choosing prices, quantities, or any other strategic variable.[1] For this interpretation to be valid, we need to assume that there is a Nash equilibrium selection in the product-market game that can be identified for every (d, μ).[2]

Assumption 1. *For all $i \in \mathcal{I}$ and all μ,*

i) $\pi^{iG}(1; \mu) > \pi^{iB}(1; \mu)$.

ii) $\pi^{i}(0; \mu) - c \geq 0$.

iii) For $\theta_i \in \{B, G\}$, $\pi_{\mu}^{i\theta_i}(1; \mu) \leq 0$ *and* $\pi_{\mu}^{i}(0; \mu) \leq 0$.

Part 1 says that the idea's profitability, ceteris paribus, is higher when the state is good than when it is bad. Part 2 establishes that firm i's profits are positive while doing R&D. Part 3 defines more intense competition as any parameter change that lowers firm i's profits, independently of the realized state and whether the innovation is implemented or not.

Example. *First, let's consider a Cournot game with linear demand $a + bQ$ and constant marginal costs c_L^i when the innovation is good, c_H^i when the innovation is bad, and c^i when no innovation is adopted, with $c_L^i < c^i < c_H^i$. Let $d^{-i} \equiv (\ldots, d^{i-1}, d^{i+1}, \ldots) \in \{0,1\}^{n-1}$, where $d^{i'} = 1$ when competitor i' adopts the more efficient technology available. Then $\pi^{i\theta_i}(d_i, \mu) = \frac{1}{b}\Big(\frac{a + \sum_{i' \neq i}(d_{i'} c_{\theta^{i'}}^{i'} + (1-d_{i'})c^{i'}}{n+1} - \frac{n}{n+1}(d_i c_{\theta^i}^i + (1-d_i)c^i)\Big)$. In this case $\mu = d^{-i}$ or $\mu = n$.*

We can consider the Perloff and Salop's (1985) model with constant marginal costs as in the Cournot example. In this case, $\mu = d^{-i}$ or μ is the number of firms. Balmaceda (2020) shows that profits are decreasing in the marginal cost of any firm since profits are log-supermodular in prices and marginal costs. If we adopt the type II extreme value distribution, μ could be the shape parameter of the distribution, which captures heterogeneity in consumer tastes.

The model assumes that the number of firms is exogenous. Thus, we are silent about the determinants of market structure (e.g., entry costs) and factors that could trigger changes in the number

[1] See, Athey and Schmutzler (2001), Schmutzler (2013) and Boone (2000, 2001) for a similar reduced-form approach.

[2] The standard approach is to assume that there is a unique locally stable equilibrium profile in the third stage that depends smoothly on investment and parameters. For instance, Milgrom and Roberts (1990) show that this holds for Bertrand's competition with differentiated goods, and Amir (1996) shows that this is the case for the Cournot game with fixed marginal costs when the inverse of the demand function is log-concave. To get uniqueness, it is usually assumed that product-market payoffs satisfy the well-known dominant diagonal condition in the corresponding strategic variable.

of competitors (e.g., mergers). Because the primary purpose of our analysis is to understand how product-market competition and R&D impact innovation, independent of what factors could explain a change in competition, we chose not to endogenize the market structure.

A firm's strategy is given by a tuple (τ^i, d^i), where $\tau^i \geq 0$ is the (possibly random) time firm i spends conducting R&D, and $d^i \in \{0, 1\}$ is the firm's decision either to implement its idea ($d^i = 1$) or to discard it ($d^i = 0$).

Belief Updating Process: When the firm conducts R&D, it privately gathers information about the idea and uses it to update its belief about its type.

To model the stochastic evolution of the belief process δ_t, we adopt the *statistical experiment* framework from (Peskir and Shiryaev, 2006, Chapter VI, §21). Specifically, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P}_\delta, \delta \in [0,1])$, where the random variable θ satisfies $\mathbb{P}(\theta = G) = 1 - \delta$ and $\mathbb{P}_\delta(\theta = B) = \delta$. As the firm conducts R&D, it privately observes the evolution of a signal process X given by

$$X_t = \theta\, t + \frac{W_t}{\sigma},$$

where W_t is a standard Brownian motion under $\mathbb{P}$, and $\sigma > 0$ determines the signal-to-noise ratio of the information generated through R&D. In this setting, a larger value of σ corresponds to a faster rate of learning, and we refer to σ as the "speed of learning" parameter.

In this setting, under Bayes' rule, the belief process $\delta_t = \mathbb{P}(\theta = B \mid \mathcal{F}_t)$ is given by

$$\delta_t = \frac{\delta}{\delta + (1-\delta)\, \mathscr{L}_t},$$

where $\mathscr{L}_t$ is the likelihood process, defined as the Radon–Nikodym derivative of $\mathbb{P}_0$ with respect to $\mathbb{P}_1$, and satisfies

$$\mathscr{L}_t = \frac{\mathrm{d}[\mathbb{P}_0|\mathcal{F}_t]}{\mathrm{d}[\mathbb{P}_1|\mathcal{F}_t]} = \exp\left(\sigma^2\left(\frac{t}{2} - X_t\right)\right),$$

and $\mathcal{F}_t$ is filtration generated by X_t.

Applying Itô's lemma, we conclude that the firm's information acquisition process causes its belief δ_t to evolve continuously over time according to the following stochastic differential equation:

$$\mathrm{d}\delta_t = \delta_t\,(1-\delta_t)\,\sigma\,\mathrm{d}B_t \qquad \text{with initial condition } \delta_0 = \delta, \tag{1}$$

where $B_t = \sigma\,(X_t - \int_0^t \delta_s\,\mathrm{d}s)$ is a standard Brownian motion with respect to $\mathcal{F}_t$.

The term $\delta_t(1-\delta_t)$ reflects the idea that new information has a smaller effect on posterior beliefs when the firm is more certain about the project's type, that is, when δ_t is close to 0 or 1. Under (1), we interpret $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ as the filtration generated by B_t, and we define $\mathbb{T}$ to denote the set of $\mathbb{F}$-stopping times.

3.2 Real-World Applications of the Model

The theoretical framework developed in this paper—where R&D is modeled as a sequential Bayesian learning process under uncertainty, innovations are non-drastic and non-exclusive, and firms may explore multiple correlated ideas—applies to several real-world settings where innovation is incremental, competitive, and strategically responsive.

A classic example is the **VHS vs. Betamax** format war. In the 1970s and 1980s, Sony's Betamax and JVC's VHS competed in the home video recorder market. The innovations were not mutually exclusive; both formats coexisted for years, with firms updating beliefs about consumer preferences and profitability through market feedback. The strategic decisions about standard-setting and licensing reflected sequential experimentation, learning, and response to competition—key features captured in our model.

In **pharmaceuticals**, particularly the development of **SSRI antidepressants** (e.g., Prozac, Zoloft, Lexapro), firms engage in extended R&D processes under clinical uncertainty. Competing laboratories often pursue similar molecules sequentially or in parallel, with innovations differing in tolerability or efficacy rather than rendering others obsolete. Clinical trials serve as information-gathering stages where firms update beliefs over time. Moreover, failures or successes in earlier compounds can inform decisions about whether to pursue related drug candidates—precisely the setting of sequential, correlated ideas.

Similarly, in the **microprocessor industry**, firms like **Intel and AMD** engage in repeated innovation cycles. Each generation of processors is developed under uncertainty about technical feasibility and market demand. Innovations are incremental and competing, with each firm observing its rival's progress and adjusting its own R&D intensity and launch timing accordingly. This dynamic reflects both the leader-follower framework and belief updating mechanisms central to our model.

In consumer electronics, **Apple and Android-based manufacturers** (e.g., Samsung, Google) provide another relevant context. Features like facial recognition, foldable displays, or custom chipsets are adopted progressively across firms, often after observing market responses to early implementations. Innovations are rarely drastic; multiple firms adopt differentiated versions of the same technological ideas, learning from each other's experimentation outcomes.

A recent example is the **development of COVID-19 vaccines**. Firms such as Pfizer-BioNTech, Moderna, and AstraZeneca pursued different platforms (mRNA, viral vector), each under significant scientific uncertainty. Firms updated beliefs during phased clinical trials and adjusted R&D strategies based on interim results, regulatory feedback, and competitor progress. The process involved multiple candidates, strategic timing, and correlated scientific approaches, matching the model's core assumptions.

Finally, the **electric vehicle battery industry** (e.g., Tesla, CATL, LG Chem) is characterized by exploration of competing chemistries (e.g., lithium iron phosphate vs. solid-state). Firms must decide whether to continue investment in a given battery technology or switch to an alternative, based on evolving data on safety, performance, and cost. As with other cases, innovations often complement rather than displace existing technologies, and firms benefit from information spillovers while adjusting R&D trajectories dynamically.

These cases illustrate the broad applicability of our framework to innovation environments where R&D is costly, uncertain, and dynamically shaped by evolving beliefs and competitive pressures. Unlike winner-take-all models, the settings above emphasize *partial substitution, sequential experimentation, and strategic timing*, features that are central to the contribution of this paper.

4 One Innovating Firm

4.1 The Firms' Optimal R&D Strategy with One Idea

In this section, we derive firm i's optimal R&D strategy. The firm's optimal strategy solves:

$$\sup_{(\tau^i,d^i)\in\mathbb{T}\times\{0,1\}} \left\{ \int_{\tau^i}^{\infty} e^{-r\,t} \left(d^i \mathbb{E}_{\delta_\tau} \pi^{i\theta}(1;\mu) + (1-d^i)\pi^i(0;\mu)\right) dt + \int_0^{\tau^i} e^{-r\,t}(\pi^i(0;\mu)-c)dt \right\} \quad (2)$$

subject to

$$\mathrm{d}\delta_t = \delta_t\,(1-\delta)\,\sigma\,\mathrm{d}B_t \quad \text{and} \quad \delta_0 = \delta,$$

where $\mathbb{E}_\delta[\cdot]$ is the conditional expectation operator given a belief δ. Specifically, $\mathbb{E}_\delta[\cdot] := (1-\delta)\,\pi^{iG} + \delta\,\pi^{iB}$.

To understand the equation above, notice that if the firm implements the idea at time τ, it gets an expected flow payoff of $\mathbb{E}_{\delta_\tau}\pi^{i\theta}$, while if the firm stops conducting R&D and discards the idea at time τ, it gets a flow payoff of $\pi^i(0;\mu)$ thereafter. Thus, the marginal benefit of stopping and discarding the idea is a flow payoff $\pi^i(0;\mu) - (\pi^i(0;\mu) - c) = c$, and the marginal benefit of stopping and implementing the idea at time τ is a flow payoff $\mathbb{E}_\delta \pi^{i\theta}(1;\mu) - (\pi^i(0;\mu) - c)$. This is the incremental value of innovation plus the R&D costs.

Observe that firm i's expected payoff can be written as

$$\begin{aligned}
&\frac{1}{r}\left(\pi^i(0;\mu) - c + \mathbb{E}_\delta\left[e^{-r\tau}\left(d^i\left(\pi^{i\theta}(1;\mu) - \pi^i(0;\mu) + c\right) + (1-d^i)\,c\right)\right]\right)\\
=&\frac{1}{r}\left(\pi^i(0;\mu) - c + \mathbb{E}_\delta\left[e^{-r\tau}\left[\mathbb{E}_{\delta_\tau}\left(d^i\;\left(\pi^{i\theta}(1;\mu) - \pi^i(0;\mu) + c\right) + (1-d^i)\,c\right)\right]\right]\right)\\
=&\frac{1}{r}\left(\pi^i(0;\mu) - c + \mathbb{E}_\delta\left[e^{-r\tau}\left(d^i\left(\mathbb{E}_{\delta_\tau}\pi^{i\theta}(1;\mu) - \pi^i(0;\mu) + c\right) + (1-d^i)\,c\right)\right]\right).
\end{aligned}$$

where the second equality follows from the martingale property. From this, it follows that for given stopping time τ, $d^{i\star} = \mathbb{I}\big(\pi^i(0;\mu) \leq \mathbb{E}_{\delta_\tau}\pi^{i\theta}(1;\mu)\big)$.

Let's define the function $V(\delta)$ as the maximum between the benefit from implementing the idea, i.e., the incremental value of innovation plus c when the belief that the idea is bad is δ, and the benefit from discarding the idea, i.e., the R&D cost saving. Thus,

$$V(\delta) =: \max\left\{\delta^i\pi^{iB}(1;\mu) + (1-\delta^i)\pi^{iG}(1;\mu) - \pi^i(0;\mu) + c, c\right\}. \tag{3}$$

This is quasi-linear in δ and independent of τ_i, which makes the problem tractable.

Given the optimal decision $d^{i\star}$, firm i's optimal stopping problem is given by

$$\mathcal{V}(\delta) = \sup_{\tau^i\in\mathbb{T}}\left\{\mathbb{E}_\delta\left[e^{-r\tau^i}V(\delta_{\tau^i})\right]\right\} \tag{4}$$

subject to

$$\mathrm{d}\delta_t = \delta_t\,(1-\delta_t)\,\sigma\,\mathrm{d}B_t \quad \text{and} \quad \delta_0 = \delta.$$

Let's denote the optimal solution to (4) by $(\tau^{i\star}, d^{i\star})$.[3]

The solution to the firm's problem involves partitioning the belief domain $[0,1]$ into a *continuation* region, where the firm conducts R&D, and an *intervention* region, where the firm stops and selects an

[3] In the rest of this section, we omit the supra-index i when there is no risk of confusion since it is the only firm with an idea to be implemented.

optimal strategy d. Thus, the optimal stopping time τ is either zero if the initial belief δ belongs to the intervention region or equal to the first exit time of the belief process δ_t from the continuation region. From the continuity of the belief process and the fact that profits are time independent, it follows that when δ belongs to the continuation region, the optimal solution is defined by a pair of thresholds $\underline{\delta}$ and $\bar{\delta}$, with $\underline{\delta} < \delta < \bar{\delta}$, such that $\tau^* = \inf\{t \geq 0\colon \delta_t \notin (\underline{\delta}, \bar{\delta})\}$.

The following is an immediate consequence of the definition of $V(\delta_\tau)$.

Lemma 1. *For all μ,*

i) If $\pi^{iB}(1;\mu) - \pi^i(0;\mu) \geq 0$, it is optimal to implement the idea immediately ($d_0 = 1$).

ii) If $\pi^{iG}(1;\mu) - \pi^i(0;\mu) < 0$, then it is optimal to discard the idea immediately ($d_0 = 0$).

According to Lemma 1, the firm i's problem admits a trivial solution in the cases considered in the Lemma. For this reason, in what follows, we will restrict profits to those satisfying the following condition.

Assumption 2. $\pi^{iG}(1;\mu) > \pi^i(0;\mu) > \pi^{iB}(1;\mu)$.

This implies that when the idea is good with probability one, it is profitable to implement it. In contrast, when it is bad with probability one, it is optimal to discard it right away. This assumption can be interpreted as the innovating firm becoming the leader when the state is good (successful innovation) and becoming the lagging firm when the state is bad (failed innovation).

We solve the optimal stopping problem (4) using a quasi-variational inequality (QVI) approach introduced by Araman and Caldentey (2022). To this end, let us define the set of continuously differentiable functions

$$\widehat{\mathcal{C}}^2 := \left\{ f \in \mathcal{C}^1[0,1] \ : \ f''(\delta) \text{ exists } \forall \delta \in [0,1] \setminus N(f) \text{ for some finite set } N(f) \subseteq [0,1] \right\} \tag{5}$$

and the operator $\mathcal{H}$ on $\widehat{\mathcal{C}}^2$

$$(\mathcal{H}f)(\delta) := \frac{1}{2}\sigma^2 \delta^2 (1-\delta)^2 f''(\delta) - r f(\delta), \qquad \text{for all } \delta \in [0,1] \setminus N(f). \tag{6}$$

Definition 1. *The function $f \in \widehat{\mathcal{C}}^2$ satisfies the quasi-variational inequalities for the firm's optimal stopping problem in (4), if for all $\delta \in [0,1] \setminus N(f)$*

$$
\begin{aligned}
& f(\delta) - V(\delta) \geq 0 \\
& (\mathcal{H} f)(\delta) \leq 0 \\
& \big(f(\delta) - V(\delta)\big)\,(\mathcal{H} f)(\delta) = 0. \qquad \square
\end{aligned}
\tag{QVI}
$$

For every solution $f \in \widehat{\mathcal{C}}^2$ of the (QVI) conditions, we associate a stopping time τ_f given by

$$\tau_f = \inf\big\{t > 0 \colon f(\delta_t) = V(\delta_t)\big\}.$$

Theorem 1. (VERIFICATION) *Let $f \in \widehat{\mathcal{C}}^2$ be a solution of (QVI). Then,*

$$f(\delta) \geq \mathcal{V}(\delta) \quad \textit{for every } \delta \in [0,1].$$

In addition, if there exists a control τ_f associated with f such that $\mathbb{E}[\tau_f] < \infty$, then τ_f is optimal and $f(\delta) = \mathcal{V}(\delta)$.

Proof: The proof of this and other results is relegated to the Appendix. $\square$

According to the previous result, at optimality, the QVI conditions partition the interval $[0,1]$ into a *continuation region* where $f(\delta) > V(\delta)$ and an *intervention region* where $f(\delta) = V(\delta)$. To find a solution, we take full advantage that the payoff function $V(\delta)$ is a piecewise linear continuous function of $\delta \in [0,1]$ and is independent of τ. Moreover, $V(\delta)$ has only two linear pieces.

In the intervention region, the third QVI condition implies that $\mathcal{V}(\delta)$ solves $(\mathcal{H}\mathcal{V})(\delta) = 0$, that is,

$$\frac{(\sigma\,\delta\,(1-\delta))^2}{2}\,\mathcal{V}''(\delta) - r\,\mathcal{V}(\delta) = 0. \tag{7}$$

The two independent solutions to this ODE are given by $F(\delta)$ and $F(1-\delta)$ with

$$F(\delta) \equiv \frac{(1-\delta)^\gamma}{\delta^{\gamma-1}} \quad \text{where } \gamma \equiv \frac{1+\sqrt{1+8\,r/\sigma^2}}{2} \tag{8}$$

and the general solution to (7) is

$$\mathcal{V}(\delta) = \beta_0\,F(\delta) + \beta_1\,F(1-\delta), \tag{9}$$

where β_0 and β_1 are the constants of integration, whose values are determined by imposing value-matching $\mathcal{V}(\delta) = 1$ and smooth-pasting condition $\mathcal{V}_\delta(\delta) = 0$ at $\delta = \bar{\delta}$.

Let's define the auxiliary function

$$\widehat{\mathcal{V}}(\delta;\bar{\delta}) \equiv \begin{cases} \frac{(\gamma-\bar{\delta})}{(2\gamma-1)}\frac{F(\delta)}{F(\bar{\delta})} + \frac{(\gamma+\bar{\delta}-1)}{(2\gamma-1)}\frac{F(1-\delta)}{F(1-\bar{\delta})} & \text{if} \quad 0 < \delta \leq \bar{\delta}^{\star}, \\ 1 & \text{if} \quad \bar{\delta}^{\star} < \delta \leq 1. \end{cases} \tag{10}$$

This corresponds to the solution to equation (7) imposing value-matching and smooth-pasting at $\bar{\delta}$ when regular profits are normalized to 1. Thus, $\widehat{\mathcal{V}}(\delta;\bar{\delta})$ is decreasing and strictly convex in $\delta \in (0,1)$ for a fixed $\bar{\delta}$, which are properties that we use to derive the next result. Also, $\widehat{\mathcal{V}}(\delta;\bar{\delta})$ increases with $\bar{\delta}$ for a given δ.

This function is fundamental to understanding the problem because it captures the benefit of conducting R&D. Its convexity indicates that the benefit of conducting R&D rises at an increasing rate with the initial belief δ_0, since a high prior belief indicates that the project has a low expected return.

The following result follows from the previous discussion.

Proposition 1. *Suppose Assumption 2 holds. Then, firm i's expected profit is given by* $\frac{1}{r}\big(\pi^i(0;\mu) - c) + \mathcal{V}(\delta)\big)$*, where*

$$\mathcal{V}(\delta) = \begin{cases} \delta\pi^{iB}(1;\mu) + (1-\delta)\pi^{iG}(1;\mu) - \pi^i(0;\mu) + c & \text{if} \quad 0 \leq \delta \leq \underline{\delta}^{\star}, \\ \widehat{\mathcal{V}}(\delta;\bar{\delta})c & \text{if} \quad \underline{\delta}^{\star} < \delta < \bar{\delta}^{\star}, \\ c & \text{if} \quad \bar{\delta}^{\star} \leq \delta \leq 1, \end{cases} \tag{11}$$

and the thresholds $\underline{\delta}^{\star}$ *and* $\bar{\delta}^{\star}$ *are determined imposing value-matching* ($\mathcal{V}(\delta) = V(\delta)$) *and smooth-pasting* ($\mathcal{V}_\delta(\delta) = V_\delta(\delta)$) *conditions at* $\delta = \underline{\delta}^{\star}$ *and* $\delta = \bar{\delta}^{\star}$*, and satisfy*

$$\underline{\delta}^{\star} < (\pi^{iG}(1;\mu) - \pi^i(0;\mu))/(\pi^{iG}(1;\mu) - \pi^{iB}(1;\mu)) < \bar{\delta}^{\star}.$$

The profit-maximizing strategy $(\tau^{\star}, d^{\star})$ *is given by*

$$\tau^{\star} = \inf\left\{t > 0 \colon \delta_t \notin (\underline{\delta}^{\star}, \bar{\delta}^{\star})\right\} \qquad \textit{and} \qquad d^{\star} = \mathbb{I}(\delta_{\tau^{\star}} \leq \underline{\delta}^{\star}).$$

The current belief is the only relevant information for firm i's decision at each instant. The trajectory of the belief is irrelevant due to the martingale nature of the updating process and the fact that the objective function is quasi-linear and independent of t. This allows us to characterize the firm i's problem as an optimal hitting time with high and low thresholds, which are time-independent.

When the posterior reaches the high threshold, it is profit-maximizing to discard the idea, whereas when it reaches the low threshold, it is profit-maximizing to implement it.

Because the firm can stop R&D at any time, the incremental profits must be large enough so that the firm is willing to keep conducting R&D instead of discarding the idea and getting the regular profits. This occurs when the firm believes the idea is bad with a probability $\delta \in (\underline{\delta}^\star, \bar{\delta}^\star)$, as its expected present value is lower than the present value of continuing to produce with the current technology and saving R&D costs. This happens when the value-matching and smooth-pasting conditions are satisfied. Otherwise, the firm could improve its expected profits (see, Figure 1).

Similarly, since firm i can implement its idea at any time, to keep the idea as an option and continue conducting R&D, it is optimal for the firm to postpone its implementation. This occurs when the firm believes the idea is bad with a probability $\delta \in (\underline{\delta}^\star, \bar{\delta}^\star))$, as its expected present value from producing with the new technology exceeds that from producing with the current technology. When the firm stops, the posterior hits the low threshold, and implements the idea, the value-matching and smooth-pasting conditions must be satisfied. Again, if this is not met, there is room for improvement (see Figure 1).

Because waiting to get the return to the innovation and saving the cost of R&D when the idea is implemented or saving the R&D cost when discarded is costly, when the prior is neither high nor low (i.e. $\underline{\delta}^\star < \delta < \bar{\delta}^\star$), the expected discounted profits upon reaching time $t \leq \tau^\star$ must be larger than the maximum between $\mathbb{E}_\delta \pi^{i\theta}(1;\mu) - (\pi^i(0;\mu) - c)$ and c to compensate for the extra cost of keep conducting R&D (see Figure 1).

We have shown that the optimal R&D strategy is constant; however, it is not memoryless in the sense that at each instant the decision to continue or stop R&D depends crucially on the information accumulated up to the instant before the decision is made.

We next provide a probabilistic characterization of the firm i's optimal R&D strategy $(\tau^\star, d^\star)$. The result is based on the dynamics of the belief process δ_t, as detailed in Equation (1), the hitting time representation of $\tau^\star$ in Proposition 1, and Dynkin's formula (see Øksendal, 2013).

Proposition 2. *Suppose $\delta \in (\underline{\delta}, \bar{\delta})$, then the optimal stopping has the moment generating function*

$$\mathbb{E}_\delta[e^{-r\,\tau}] = \frac{(F(1-\bar{\delta}) - F(1-\underline{\delta}))\,F(\delta) + (F(\underline{\delta}) - F(\bar{\delta}))\,F(1-\delta)}{F(\underline{\delta})\,F(1-\bar{\delta}) - F(\bar{\delta})\,F(1-\underline{\delta})}$$

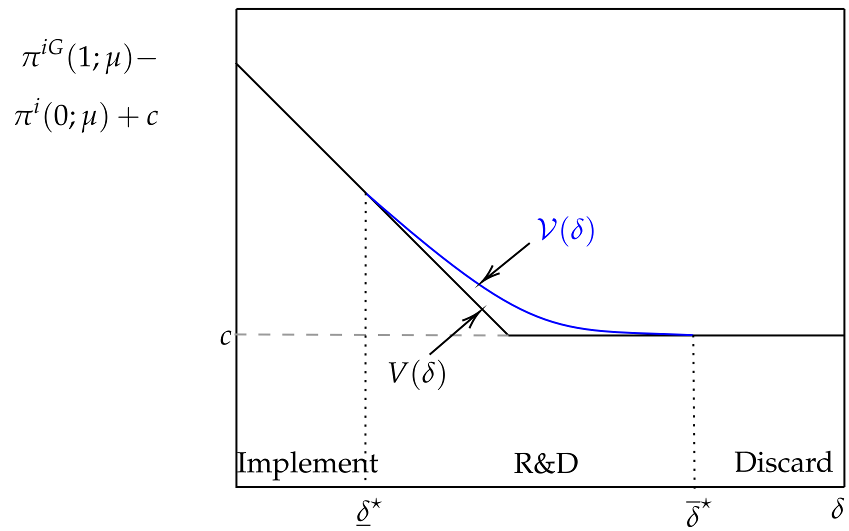

Figure 1: firm i's expected discounted payoff $\mathcal{V}(\delta)$ as a function of the belief δ. The range of beliefs is partition into three regions: (i) for $\delta \in [0, \underline{\delta}^{\star}]$ the firm implements the idea, (ii) for $\delta \in (\underline{\delta}^{\star}, \bar{\delta}^{\star})$ the firm conducts R&D and (iii) for $\delta \in [\bar{\delta}^{\star}, 1]$ the firm discard the idea.

$$\textit{and satisfies } \mathbb{E}_{\delta}\left[e^{-r\,\tau}\,\mathbb{I}(\delta_{\tau} = \underline{\delta})\right] = \frac{F(\delta)\,F(1-\bar{\delta}) - F(\bar{\delta})\,F(1-\delta)}{F(\underline{\delta})\,F(1-\bar{\delta}) - F(\bar{\delta})\,F(1-\underline{\delta})}, \textit{ where } F(\delta) \textit{ is given in } (8).$$

The expected amount of time firm i spends conducting R&D is equal to

$$\mathbb{E}_{\delta}[\tau^{\star}] = \left(\frac{\bar{\delta}^{\star}-\delta}{\bar{\delta}^{\star}-\underline{\delta}^{\star}}\right) g(\underline{\delta}^{\star}) + \left(\frac{\delta-\underline{\delta}^{\star}}{\bar{\delta}^{\star}-\underline{\delta}^{\star}}\right) g(\bar{\delta}^{\star}) - g(\delta), \quad \textit{where} \quad g(\delta) = \frac{2\,(1-2\delta)}{\sigma^{2}} \ln\left(\frac{1-\delta}{\delta}\right).$$

The probability that firm i implements its innovation and the probability that it does not are given by

$$\mathbb{P}_{\delta}(d^{\star}=1) = \frac{\bar{\delta}^{\star}-\delta}{\bar{\delta}^{\star}-\underline{\delta}^{\star}} \qquad \textit{and} \qquad \mathbb{P}_{\delta}(d^{\star}=0) = \frac{\delta-\underline{\delta}^{\star}}{\bar{\delta}^{\star}-\underline{\delta}^{\star}}, \quad \textit{respectively.}$$

4.2 Comparative Statics: R&D and Competition

In this subsection, we study the effect of competition on the three different but related measures of R&D. One is the difference between the high and the low threshold, $\bar{\delta}^{*} - \underline{\delta}^{*}$. Another is the probability that the idea is implemented. The other is the expected amount of time the firm spends conducting R&D.

To get the optimal $(\underline{\delta}, \bar{\delta})$, we use the value-matching and smooth-pasting at $\delta = \underline{\delta}$. This means that $(\underline{\delta}, \bar{\delta})$ solve the following equations

$$\widehat{\mathcal{V}}(\underline{\delta};\bar{\delta})c = \underline{\delta}\pi^{iB}(1;\mu) + (1-\underline{\delta})\pi^{iG}(1;\mu) - \pi^{i}(0;\mu) + c \tag{12}$$

and

$$\widehat{\mathcal{V}}_{\delta}(\underline{\delta};\bar{\delta})c = \pi^{iB}(1;\mu) - \pi^{iG}(1;\mu) \tag{13}$$

The value matching captures Arrow (1962). He showed that firms' incentives to invest in R&D are driven by the incremental value they obtain from an innovation. For the instant t, the firm ceases conducting R&D when its instant return from R&D equals the expected incremental value of innovation, which is the innovation's expected profits, evaluated at the current belief, minus the profits that the firm earns while conducting R&D (regular profits minus R&D costs). Because $\widehat{\mathcal{V}}(\underline{\delta};\bar{\delta})$ rises with $\bar{\delta}$, the larger the incremental value of innovation, ceteris paribus, the higher is $\bar{\delta}$, i.e., the smaller the rejection region.

Smooth-pasting indicates that the firm stops doing R&D and implements the innovation whenever the change in the incremental value with the belief that the state is bad equals the marginal value gain from improving its information by spending a instant more conducting R&D. Because $\widehat{\mathcal{V}}(\underline{\delta};\bar{\delta})$ falls with δ, the higher the marginal increase in the incremental value $\pi^{iG}(1;\mu) - \pi^{iB}(1;\mu)$, the stronger the incentives to implement the innovation. Holding $\bar{\delta}$ constant, this means a larger $\underline{\delta}$, i.e., the larger the acceptance region.

The first measure of competitiveness we consider is the regular profits, i.e., profits the firm makes while conducting R&D or when the idea is discarded, which is given by $\pi^{i}(0;\mu)$. For instance, if competitors are identical and compete in prices, firm's profits should be zero when it does not innovate. If they compete in prices with differentiated goods, this should be positive and large when goods are highly differentiated. Thus, we assume that the more competitive the market under the current technology, the smaller the regular profits.

As argued by Arrow (1962), when a firm innovates, it cannibalizes its profits. Thus, when the firm innovates, it replaces $\pi^{i}(0;\mu)$, i.e., the profit the firm makes with the current technology, with $\pi^{i\theta_i}(1;\mu)$, i.e., the profit the firm makes with the new technology. The larger the regular profits, the lower the incremental value of the innovation. Thus, the Arrow's replacement effect implies that R&D should fall as $\pi^{i}(0;\mu)$ rises.

Proposition 3 (*Arrow's Replacement Effect*)**.**

i) $\bar{\delta}^\star$, $\underline{\delta}^\star$, *and* $\bar{\delta}^\star - \underline{\delta}^\star$ *fall with* $\pi^i(0;\mu)$.

ii) The probability that the idea is implemented falls with $\pi^i(0;\mu)$.

iii) $\mathbb{E}_\delta[\tau^\star]$ *falls with* $\pi^i(0;\mu)$.

An increase in regular profits, i.e., the less competitive the pre-innovation market, decreases the incremental value of innovation. On the one hand, this implies that the cost of discarding the idea falls. Thus, $\bar{\delta}$ falls. On the other hand, the cost of conducting R&D is smaller since, while doing it, the firm gets higher profits. Thus, $\underline{\delta}$ falls. Because $\hat{\mathcal{V}}$ falls with $\underline{\delta}$, rises with $\bar{\delta}$, and $\hat{\mathcal{V}}_{\underline{\delta}\bar{\delta}} < 0$, the first effect dominates and $\bar{\delta} - \underline{\delta}$ falls. This implies that the innovation is implemented less often and, on average, the firm conducts R&D for a shorter expected time. A consequence of this is that the decision is made, on average, with less information regarding the quality of the idea being researched.

This confirms the existence of the Arrow's replacement effect and reveals a new phenomenon: the quality of information gathered also decreases on average as regular profits rise.

Next, we examine how our R&D measures are influenced by competition intensity, as measured by μ. This exercise differs from the preceding one in that competition intensity lowers both regular profits and those from innovation. Consequently, more intense competition may either increase or decrease the incremental value of innovation. Thus, more intense competition results not only in a replacement effect but also in a reducing-profitability effect. Furthermore, an increase in μ changes the marginal change in the incremental value $\pi^{iB} - \pi^{iG}$, which modifies the firm's optimal choice of $\underline{\delta}$, holding $\bar{\delta}$ constant.

In the rest of the paper, we will focus on the following case

Assumption 3. *For all* μ,

$$\pi^{iG}_\mu \geq \pi^i_\mu \geq \pi^{iB}_\mu.$$

This indicates that competition decreases the marginal loss in the incremental value due to an increase in the belief the state is bad. Additionally, regular profits are less affected by competition intensity than those from a failed technological innovation or the introduction of a poorly developed product. In other words, the more intense the competition, the less affected the leadin firm is, and the more affected the lagging firm is. The former means that, holding $\bar{\delta}$ constant, $\underline{\delta}$ rises, i.e., the firm is more inclined to implement the innovation, since in a more competitive environment the profit loss in the good state is smaller than that in the bad state. In the case of linear Cournot, this holds when $\mu = n$ and holds with equality when $\mu = d_{-i}$.

Proposition 4 (Competition Intensity)**.** *Suppose that Assumption 3 holds.*

i) If

$$\underline{\delta}^\star \pi^{iB}_\mu + (1-\underline{\delta}^\star)\pi^{iG}_\mu - \pi^i_\mu \leq 0, \tag{14}$$

then $(\bar{\delta}^\star, \underline{\delta}^\star)$ and $\mathbb{P}_{\bar{\delta}}(d^\star = 1)$ fall with μ. Otherwise $\bar{\delta}^\star$ rises with μ, $\underline{\delta}^\star$ falls with it, and there is threshold for the prior belief $\hat{\delta}_1 \in (\underline{\delta}^\star, \bar{\delta}^\star)$ such that $\mathbb{P}_{\bar{\delta}}(d^\star = 1)$ falls with μ whenever $\delta_1 \leq \hat{\delta}_1$ and rises otherwise.

ii) $\bar{\delta}^\star - \underline{\delta}^\star$ rises with μ.

iii) $\mathbb{E}_\delta[\tau^\star]$ rises with μ.

The following observations are key to understand this result: $\widehat{\mathcal{V}}(\underline{\delta};\bar{\delta})$ rises with $\bar{\delta}$ and falls with $\underline{\delta}$; $\widehat{\mathcal{V}}(\underline{\delta};\bar{\delta})$ is convex in $\underline{\delta}$ and submodular in $(\underline{\delta},\bar{\delta})$; and $\widehat{\mathcal{V}}_{\bar{\delta}\underline{\delta}}(\underline{\delta};\bar{\delta}) + \widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta};\bar{\delta}) \geq 0$.

The first says that when a more intense competition intensity rises the incremental value of innovation, the firm is less likely to discard the project, i.e., holding $\underline{\delta}$ constant, $\bar{\delta}$ rises, and more likely to implement the project, i.e., holding $\bar{\delta}$ constant, $\underline{\delta}$ rises, since $\pi^{iB} - \pi^{iG}$ falls.

The second means that the marginal benefit from carrying R&D, holding $\bar{\delta}$ constant, decreases at a decreasing rate with $\underline{\delta}$, and the marginal benefit of increasing the low threshold below which the idea is implemented decreases with the high threshold above which the idea is rejected ($\bar{\delta}$).

The third part says that the impact of an increase in $\underline{\delta}$ on the marginal return to information at the threshold where the innovation is implemented dominates the effects of an increase in $\bar{\delta}$ on that. This is referred to as the dominant diagonal condition due to its parallelism with the standard dominant diagonal condition.

On the one hand, because increased competition intensity lowers regular profits, the replacement effect result derived in Proposition 3, implies, ceteris paribus, that the firm implements the idea more frequently and, on average, increases the time conducting R&D. On the other hand, an increased competition intensity decreases the expected profits from implementing the innovation. This implies that discarding the innovation is more attractive and implementing it is less appealing.

Provided that the marginal change in the incremental value with the belief that the state is bad falls with competition, when the expected incremental value of innovation decreases with the intensity of competition, the profit-decreasing effect dominates the replacement effect. Thus, the high and low thresholds and their differences fall, as does the probability that the innovation will be implemented. This occurs because of the dominant diagonal condition.

When the incremental value of innovation rises with competition intensity, countervailing forces are at work. On one hand, $\pi^{iB} - \pi^{iG}$ falls, which, ceteris paribus, provides the firm with stronger incentives to implement its innovation. On the other hand, the incremental value of the innovation rises, which implies the firm tolerates more bad news before discarding the project. The proposition proves that the first effect prevails when the prior is lower than a specified threshold, as the expected benefit of learning good news is higher in this scenario.

The time spent conducting R&D rises as competition intensity increases due to the convexity of the option value of it. Convexity implies that the marginal benefit of acquiring information for an instant rises as $\underline{\delta}$ falls. Thus, the firm's marginal benefit from doing R&D rises as it sets a lower threshold to implement the project. Because $\widehat{\mathcal{V}}(\underline{\delta};\bar{\delta})$ is submodular in $(\underline{\delta},\bar{\delta})$, the decrease in $\underline{\delta}$ pushes the return to discard the idea more often down.

Thus, a-priori, the relationship between competition and R&D when this is measured by the probability of implementing the innovation is ambiguous and depends on whether more competition increases or decreases the incremental value of the idea evaluated at the optimal hitting times, as shown by Arrow (1962). However, it also depends on how the marginal change in the incremental value with the belief that the state is bad varies with the intensity of competition. Under the assumption that this falls with competition, when the incremental value decreases due to competition, the relationship is negative. In contrast, when the incremental value of the innovation more than compensates for the value loss of regular profits, the relationship is positive when the probability that the idea is bad is large and negative otherwise. When R&D is measured by the expected time conducting R&D, the relationship is always positive, i.e., as competition intensity rises, the firm conducts R&D for a longer expected time. Thus, on average, as competition rises, the firm wishes to make a more inform decision.

The empirical evidence indirectly confirms the theoretical ambiguity in the relationship between competition and innovation. Studies such as Baily et al. (1995), Blundell et al. (1995), and Nickell (1996) find a positive association between competition and innovation, while Aghion et al. (2005)

document an inverted-U relationship in U.K. manufacturing. In contrast, Hashmi (2013) reports a negative relationship in U.S. data and attributes the difference to a lower degree of technological neck-and-neck competition compared to the U.K.

Other studies highlight the heterogeneity of innovation responses. Beneito et al. (2015) show that product substitutability and entry costs raise process but lower product innovation, while market size boosts both. Kretschmer et al. (2012), using data from the French automotive sector, find that increased competition lowers process but raises product innovation, emphasizing the role of scale effects. Goettler and Gordon (2011), analyzing AMD–Intel rivalry, find that while competition slowed innovation, it enhanced consumer welfare via lower prices. Finally, Hashmi and Biesebroeck (2016) finds that in the automobile industry, entry—particularly by high-quality firms—reduces innovation within individual firms but increases industry-wide innovation due to increased diversity.

Overall, these findings highlight that the impact of competition on innovation is context-dependent, shaped by market structure and innovation type, and the measure of competition intensity used.

4.3 One Innovating Firm with Two Ideas

In this subsection, we extend our previous analysis by considering the case where the firm, after discarding the initial idea, can explore a second idea whose stochastic process is correlated with that of the initial idea. The goal is to study how having correlated ideas that can be explored sequentially affects the firm's incentives to conduct R&D in the earlier idea. This fits well the case in which R&D reveals that the path being investigated was not as good as previously thought, and thus it is worth abandoning it and following a different path, although the path followed provides relevant information.

Let's assume there are two ideas, denoted by a and b. First, the firm decides whether to discard, implement, or carry R&D on idea a. After a decision is made, the firm can carry R&D in idea b whenever it chooses not to implement idea a.[4]

Let $j \in \{a, b\}$, $\theta_j \in \{B, G\}$, and $d_j = 1$ when idea j is implemented, and $d_j = 0$ when rejected. Profits are given by $\pi^{i\theta_j}(d_j; \mu)$ and satisfy the following:

[4] Implicitly, we assume that the cost of running two parallel R&D processes is infinite. This could be due to limited financial resources, a limited number of scientists or labs, or legal and technical constraints.

Assumption 4. *For* $(d_a, \theta_a) = (d_b, \theta_b)$, $\pi^{i\theta_a}(d_a; \mu) = \pi^{i\theta_b}(d_b; \mu)$, *and* $\pi^{iG}(1; \mu) > \pi^{i}(0; \mu) \geq \pi^{iB}(1; \mu)$.

Thus, the profits from ideas a and b are identical in each state.[5] However, they differ in their stochastic process as specified below. Thus, we focus only on the information aspect of the learning dynamics when there are two ideas, and not on the impact of differences in profitability between them.

Let $\eta_t = \mathbb{P}(\eta = 1 \mid \mathcal{G}_t)$ denote the firm's belief that the idea b is bad after conducting due diligence for t time units. The SDE governing the R&D in idea a is given by that in equation (1), whereas the SDE governing the R&D in idea b is as follows

$$d\eta_t = \eta_t\,(1-\eta_t)\,\mathrm{d}A_t \quad \text{and} \quad A_t = \sigma\,\rho B_t + \sigma_b\,\sqrt{1-\rho^2}C_t, \text{ and } \eta_0 = \eta \text{ otherwise.} \tag{15}$$

where B_t and C_t are two independent Brownian motions. Thus, A_t and B_t are correlated with correlation coefficient $\rho \in [-1, 1]$ and $\mathcal{G}_t$ denote the usual filtration generated by A_t. Let $\mathbb{T}$ be the set of stopping times regarding $\mathbb{G}$. We will naturally require $\tau_b \in \mathbb{T}$ and $d_b \in \mathcal{G}_\tau$.

Let's define the function

$$V(\eta) \equiv \max\left\{\eta\pi^{iB}(1;\mu) + (1-\eta)\pi^{iG}(1;\mu) - \pi^{i}(0;\mu) + c, c\right\}.$$

The firm's optimal stopping problem when researching idea b is given by

$$\mathcal{V}(\eta) =: \sup_{\tau_b \in \mathbb{T}}\left\{\mathbb{E}_\eta\left[e^{-r\tau_b}V(\eta_{\tau_b})\right]\right\}$$

subject to

$$\mathrm{d}\eta_t = \eta_t\,(1-\eta_t)\,(\sigma\,\rho B_t + \sigma_b\,\sqrt{1-\rho^2}C_t) \quad \text{and} \quad \eta_0 = \eta.$$

According to the verification theorem, at optimality, the QVI conditions partition the interval $[0, 1]$ into a *continuation region* where $f(\eta) > V(\eta)$ and an *intervention region* where $f(\eta) = V(\eta)$. To find a solution, as in the preceding subsection, we take advantage of the payoff function $V(\eta)$ being a piecewise linear continuous function of $\eta \in [0, 1]$ with only two linear pieces.

In the intervention region, the third QVI condition implies that $\mathcal{V}(\eta)$ solves $(\mathcal{H}\mathcal{V})(\eta) = 0$, that is,

$$\frac{(\sigma(\rho)\,\eta\,(1-\eta))^2}{2}\,\mathcal{V}''(\eta) - r\,\mathcal{V}(\eta) = 0, \tag{16}$$

[5]This assumption is not needed, but it simplifies the notation and the algebra.

where $\sigma(\rho) \equiv \sigma^2\rho^2 + \sigma_b^2(1-\rho^2)$.

Thus, this problem has the same structure as the one we solved in the last section. The solution, is (d_b, τ_b) and a cutoff strategy $(\underline{\eta}, \bar{\eta})$ with parameter $\sigma(\rho)$ instead of σ. Observe that $(\underline{\eta}, \bar{\eta})$ satisfies the value matching condition $\mathcal{V}(\bar{\eta})c = 1$ and the smooth-pasting condition is $\mathcal{V}_\eta(\bar{\eta})c = 0$. The solution to these equations is denoted by $\widehat{\mathcal{V}}(\eta; \bar{\eta}, \rho)$.

The equilibrium profit is given by $\frac{1}{r}\Pi(\eta; d_a, \theta_a)$, with $\Pi(\eta; d_a, \theta_a) \equiv (1-d_a)(\pi^i(0;\mu) - c + \mathcal{V}(\eta;\rho)) + d_a \pi^{i\theta_a}(1;\mu)$, where

$$\mathcal{V}(\eta;\rho) = \begin{cases} \eta\pi^{iB}(1;\mu) + (1-\eta)\pi^{iG}(1;\mu) - \pi^i(0;\mu) + c & \text{if} \quad 0 \leq \eta \leq \underline{\eta}^\star, \\ \widehat{\mathcal{V}}(\eta;\bar{\eta},\rho)c & \text{if} \quad \underline{\eta}^\star < \eta < \bar{\eta}^\star, \\ c & \text{if} \quad \bar{\eta}^\star \leq \eta \leq 1, \end{cases} \tag{17}$$

and

$$\widehat{\mathcal{V}}(\eta;\bar{\eta},\rho) \equiv \begin{cases} \frac{(\gamma(\rho)-\bar{\eta})}{(2\gamma(\rho)-1)}\frac{F(\eta)}{F(\bar{\eta})} + \frac{(\gamma(\rho)+\bar{\eta}-1)}{(2\gamma(\rho)-1)}\frac{F(1-\eta)}{F(1-\bar{\eta})} & \text{if} \quad 0 < \eta \leq \bar{\eta}, \\ 1 & \text{if} \quad \bar{\eta} < \eta \leq 1. \end{cases} \tag{18}$$

The profit-maximizing strategy $(\tau_b^\star, d_b^\star)$ is given by

$$\tau_b^\star = \inf\{t > 0 \colon \eta_t \notin (\underline{\eta}^\star, \bar{\eta}^\star)\} \qquad \text{and} \qquad d_b^\star = \mathbb{I}(\eta_{\tau_b^\star} \leq \underline{\eta}^\star).$$

The intuition behind this result is identical to that for the case where the firm has only one idea. Therefore, we will not discuss it further here for the sake of brevity.

The next proposition follows from Proposition 4 and because $\widehat{\mathcal{V}}(\eta;\bar{\eta},\rho)$ rises with the correlation coefficient ρ.

Proposition 5. *If $\sigma^2 - \sigma_b^2 \geq 0$, $\bar{\eta}^\star$, $\underline{\eta}^\star$, $\bar{\eta}^\star - \underline{\eta}^\star$, and $\mathbb{P}_\eta(d_b = 1)$ rise, and $\mathbb{E}[\tau_b^\star]$ falls with ρ.*

This says that if the signal-to-noise ratio of the idea b's process is smaller than that of the idea a's process, then as the correlation between the processes rises, the speed of learning, as measured by $\gamma(\rho)$, rises, which means that less time, on average, should be required to gather the same information quality. Consequently, the high and low thresholds and their difference increase since each experimental instant brings more information. Thus, the firm is more likely to implement the idea and less likely to reject it, and spends, on average, less time experimenting. The opposite occurs if learning takes place at a faster pace under the latter idea than under the earlier one.

The firm's payoff in period 0 is

$$\mathbb{E}_{\tau_b,\eta,\delta}\left[e^{-r\,\tau_a}\Pi(\eta;d_a,\theta_a)+\int_0^{\tau_a}e^{-r\,t}\big(\pi^i(0;\mu)-c\big)dt\right].$$

From time zero to the time the firm stops exploring idea a, the firm gets its regular profits minus the cost of R&D. After that, if the firm implements the idea, it gets the benefit of implementing innovation a. If it does not implement it, the firm gets the regular profits minus the R&D costs while conducting R&D in idea b. Finally, after the firm stops conducting R&D in idea b, it benefits from idea b when implemented, and from regular profits otherwise.

Let's define the following function

$$V(\delta;\rho)=\max\left\{\delta\pi^{iB}(1;\mu)+(1-\delta)\pi^{iG}(1;\mu)-\pi^i(0;\mu)+c,\mathcal{V}(\eta;\rho)\right\}.$$

Given the optimal decision $d_a^\star$, firm i's optimal stopping problem at time zero is given by

$$\mathcal{V}(\delta;\rho)=\sup_{\tau_a\in\mathbb{T}}\left\{\mathbb{E}_\delta\left[e^{-r\tau_a}V(\delta_{\tau_a};\eta,\rho)\right]\right\} \tag{19}$$

subject to

$$\mathrm{d}\delta_t=\delta_t\,(1-\delta_t)\,\sigma\,\mathrm{d}B_t\quad\text{and}\quad\delta_0=\delta.$$

Solving the SDE that result from the QVI conditions by imposing smooth-pasting and value-matching at $\delta=\bar{\delta}$, gives rise to the following auxiliary function

$$\widehat{\mathcal{V}}(\delta;\bar{\delta})\equiv\begin{cases}(\gamma-\bar{\delta})\,\frac{F(\delta)}{F(\bar{\delta})}+(\gamma+\bar{\delta}-1)\,\frac{F(1-\delta)}{F(1-\bar{\delta})} & \text{if}\quad 0\le\delta<\bar{\delta}\\ 1 & \text{if}\quad \bar{\delta}^\star\le\delta\le 1\end{cases} \tag{20}$$

This is decreasing and strictly convex in $\delta\in(0,1)$ for a fixed $\bar{\delta}$, which are properties that we use to derive the next result. Also, $\widehat{\mathcal{V}}(\delta;\bar{\delta})$ increases with $\bar{\delta}$ for any given δ.

The following result follows from the previous discussion and Proposition 1.

Proposition 6. *Suppose Assumption 2 holds. Then, firm i's expected profit is given by* $\frac{1}{r}(\pi^i(0;\mu)-c+\mathcal{V}(\delta;\eta))$, *where*

$$\mathcal{V}(\delta;\eta,\rho)=\begin{cases}\delta\pi^{iB}(1;\mu)+(1-\delta)\pi^{iG}(1;\mu)-\pi^i(0;\mu)+c & \text{if}\quad 0\le\delta\le\underline{\delta}^\star,\\ \widehat{\mathcal{V}}(\delta;\bar{\delta})\mathcal{V}(\eta;\rho) & \text{if}\quad \underline{\delta}^\star<\delta<\bar{\delta}^\star,\\ \mathcal{V}(\eta;\rho) & \text{if}\quad \bar{\delta}^\star\le\delta\le 1.\end{cases} \tag{21}$$

and the thresholds $\underline{\delta}^\star$ and $\bar{\delta}^\star$ are determined imposing value-matching and smooth-pasting conditions at $\delta = \underline{\delta}^\star$ and $\delta = \bar{\delta}^\star$. The profit-maximizing strategy $(\tau_a^\star, d_a^\star)$ is given by

$$\tau_a^\star = \inf\left\{t > 0\colon \delta_t \notin (\underline{\delta}^\star, \bar{\delta}^\star)\right\} \qquad \textit{and} \qquad d_a^\star = \mathbb{I}(\delta_{\tau_a^\star} \leq \underline{\delta}^\star).$$

The value-matching and smooth-pasting conditions that determine $(\underline{\delta}, \bar{\delta})$ are given by

$$\widehat{\mathcal{V}}(\underline{\delta}; \bar{\delta})\mathcal{V}(\eta;\rho) = \delta\pi^{iB}(1;\mu) + (1-\delta)\pi^{iG}(1;\mu) - \pi^{i}(0;\mu) + c \tag{22}$$

and

$$\widehat{\mathcal{V}}_\delta(\underline{\delta}; \bar{\delta})\mathcal{V}(\eta;\rho) = \pi^{iB}(1;\mu) - \pi^{iG}(1;\mu), \tag{23}$$

where is given by equation (17).

Because $\mathcal{V}(\eta;\rho)$ rises with ρ whenever $\sigma^2 - \sigma_b^2 \geq 0$, the opportunity cost of experimenting for a longer period of time falls with ρ, and that of implementing idea rises with ρ. Thus, we deduce the next result from this and Proposition 4.

Proposition 7. *Suppose that $\underline{\eta}^\star < \eta < \bar{\eta}^\star$. If $\sigma^2 - \sigma_b^2 \geq 0$, $\bar{\delta}^\star$, $\underline{\delta}^\star$, $\bar{\delta}^\star - \underline{\delta}^\star$, and $\mathbb{P}(d^\star = 1)$ fall and $\mathbb{E}[\tau_a^\star]$ rises with ρ.*

Thus, having a pool of ideas whose learning dynamics are correlated induces the firm to increase its expected time conducting R&D and to implement its first innovation less often when the speed of learning is higher for the earlier idea. This happens because as the correlation rises, the firm's expected profits of the second idea are higher and the information gathered by conducting R&D in the first idea is more informative when screening the second one. Thus, having correlated ideas that can be researched sequentially changes the earlier R&D strategy in the direction of intensifying it when the earlier idea provides more information per unit of time than the later idea.

5 Two Innovating Firms with One Idea Each

In this section, we discuss the case in which two firms, denoted by 1 and 2, each have an idea they want to implement, but before doing so, each may engage in R&D. This is done sequentially. We call the first to conduct R&D, the leader and the second, the follower. Furthermore, we assume that there are no patents and no leapfrogging. Thus, a successful innovation by the follower does not render the leader's technology obsolete, nor does an innovation by the leader fully preempt the innovation by the follower. However, successful innovations decrease the competitor's profitability, while failed innovations have the opposite effect. Thus, a priori, innovation by the leader may induce the follower to respond more aggressively or more submissively.

We assume that firm 1 (the leader) engages in R&D first and firm 2 (the follower) does so after observing firm 1's decisions and the outcome of its innovation when implemented, i.e., if the innovation was good or bad.[6]

Let $\theta = (\theta_1, \theta_2)$ and $d_i = 1$ when firm i implements its idea, and $d_i = 0$ when discarded. Profits are given by $\pi^{i\theta}(d_i, d_{i'}; \mu)$ and satisfy the following: $\pi^{i\theta}(0, d_{i'}; \mu) = \pi^{i\theta_{i'}}(0, d_{i'}; \mu)$ for $\theta_{i'} \in \{B, G\}$, $\pi^{i\theta}(d_i, 0; \mu) = \pi^{i\theta_i}(d_i, 0; \mu)$ for $\theta_i \in \{B, G\}$, and $\pi^{i\theta}(0, 0; \mu) = \pi^{i}(0, 0; \mu)$ for all $\theta \in \{B, G\}^2$. In short, the state matters only when the corresponding innovation is implemented.

We assume the following.

Assumption 5. *For $i \in \{1, 2\}$,*

i) For $\theta_i \in \{B, G\}$ and $d_i \in \{0, 1\}$, $\pi^{i\theta_i B}(d_i, 1; \mu) \geq \pi^{i\theta_i}(d_i, 0; \mu) \geq \pi^{i\theta_i G}(d_i, 1; \mu)$.

ii) For $\theta_{-i} \in \{B, G\}$, $\pi^{iG\theta_{-i}}(1, 1; \mu) - \pi^{iG}(1, 0; \mu) \geq \pi^{i\theta_{-i}}(0, 1; \mu) - \pi^{i}(0, 0; \mu) \geq \pi^{iB\theta_{-i}}(1, 1; \mu) - \pi^{iB}(1, 0; \mu)$.

iii) $\pi^{iG}(0, 1; \mu) > 0$.

Part i) establishes that firm i has negative externalities on the competitor when it implements its innovation, and the state is good, and positive externalities when the state is bad.[7] Externalities can arise from technology spillovers, knowledge sharing, and/or incomplete appropriability, which may increase/decrease the productivity of other firms operating in similar technology areas. Alternatively, strategic spillovers may reflect product-market interactions that create an indirect link between firms' investment decisions through their anticipated impact on product-market competition.

[6] We can consider the follower to be an entrant, who enters after the leader has made its decision and the state has realized. However, in this case μ cannot be the number of firms.

[7] Externalities have been discussed in works such as Bloom et al. (2013), López and Vives (2016), and d'Aspremont and Jacquemin (1988).

Part ii) is the equivalent of assumption 3 when $\mu = d_{-i}$. This assumes that when a competitor innovates, a firm's profits fall less when its innovation is successful than when it is not, regardless of whether the competitor's innovation was successful or not.

Part iii) establishes that when firm i's competitor succeeds at innovating and firm i discards its innovation, its profits are positive. This ensures that the expected profits when discarding the innovation are positive for any prior belief about the potential success of the innovation. Otherwise, there could be priors (δ_1, δ_2) such that firm i may prefer to exit the market.

Firm 2's objective function is

$$V(\delta_2; d_1, \theta_1) =: \max\left\{\delta_2 \pi^{2B\theta_1}(1, d_1; \mu) + (1-\delta_2)\pi^{2G\theta_1}(1, d_1; \mu) - \pi^{2\theta_1}(0, d_1; \mu) + c, c\right\}. \tag{24}$$

and its optimal stopping problem for the leader's first-period state $\theta_1 \in \{B, G\}$ and decision $d_1 \in \{0, 1\}$ is given by

$$\mathcal{V}(\delta_2) =: \sup_{\tau_2 \in \mathbb{T}} \left\{\mathbb{E}_\eta\left[e^{-r\tau_2} V(\delta_{2\tau_2}; d_1, \theta_1)\right]\right\}$$

subject to

$$\mathrm{d}\delta_{2t} = \delta_{2t}\,(1-\delta_{2t})\,\sigma\, A_t \quad \text{and} \quad \delta_{20} = \delta_2.$$

Let $(\underline{\delta}_2(d_1, \theta_1), \bar{\delta}_2(d_1, \theta_1), \tau_2(d_1, \theta_1), d_2(d_1, \theta_1))$ be firm 2's optimal thresholds, optimal stopping time, and optimal decision when firm 1's decision is d_1 and the realized state is θ_1, respectively, where

$$\tau_2(d_1, \theta_1) = \inf\left\{t > 0 \colon \delta_{2t} \notin (\underline{\delta}_2(d_1, \theta_1), \bar{\delta}_2(d_1, \theta_1))\right\} \qquad \text{and} \qquad d_2(d_1, \theta_1) = \mathbb{I}(\delta_{2\tau_2(d_1,\theta_1)} \leq \underline{\delta}_2(d_1, \theta_1)),$$

and

$$\underline{\delta}_2(d_1, \theta_1) < (\pi^{2G\theta_1}(1, d_1; \mu) - \pi^{2\theta_1}(0, d_1; \mu)) / (\pi^{2G\theta_1}(1, d_1; \mu) - \pi^{2B\theta_1}(1, d_1; \mu)) < \bar{\delta}_2(d_1, \theta_1).$$

For each (θ_1, d_1), firm 2's optimization problem is the same as the one firm i faces when it is the only innovating firm with one idea. Thus, the results derived in Section 4 apply directly to firm 2's optimal-stopping problem for each possible pair (d_1, θ_1).

The value-matching and smooth-pasting conditions $\widehat{\mathcal{V}}(\bar{\delta}_2) = 1$ and $\widehat{\mathcal{V}}_{\delta_2}(\bar{\delta}_2) = 0$ determine the integration constants that solve the SDE that arises from the followers' optimal stopping problem. Furthermore, the value-matching and smooth-pasting at $(\underline{\delta}_2(d_1, \theta_1), \bar{\delta}_2(d_1, \theta_1))$ are given by

$$\widehat{\mathcal{V}}(\underline{\delta}_2(d_1, \theta_1); \bar{\delta}_2(d_1, \theta_1))c = \underline{\delta}_2 \pi^{2B\theta_1}(1, d_1; \mu) + (1-\underline{\delta}_2)\pi^{2G\theta_1}(1, d_1; \mu) - \pi^{2\theta_1}(0, d_1; \mu) + c$$

and

$$\widehat{\mathcal{V}}_\delta(\underline{\delta}_2(d_1,\theta_1);\bar{\delta}_2(d_1,\theta_1))c = \pi^{2B\theta_1}(1,d_1;\mu) - \pi^{2G\theta_1}(1,d_1;\mu).$$

The value-matching and smooth-pasting conditions determine how the follower's optimal R&D strategy responds to the leader's innovation. Clearly, if the leader's innovation increases the follower's incremental value of innovation, the follower responds by decreasing the rejection region. Because of Assumption 5 part ii, the change in the incremental value with δ_t falls with the leader's innovation, which means that the follower chooses a larger acceptance region. If it falls more when the leader's state is bad than when it is good, the acceptance region is larger when the leader's innovation is bad than when it is good. This happens when $\pi^{2\theta}(1,1;\mu)$ is submodular in θ and the opposite when it is supermodular in θ. Thus, there are counterweighing forces when the leader's state is good. Also, when the incremental value decreases and the state is bad.

We deduce the following result from this and Propositions 3 and 4.

Proposition 8.

i) If $\pi^{2\theta}(1,1;\mu)$ is submodular in θ and

$$\begin{aligned}&\underline{\delta}_2\pi^{2BG}(1,1;\mu) + (1-\underline{\delta}_2)\pi^{2GG}(1,1;\mu) - \pi^{2G}(0,1;\mu) \geq \\ &\underline{\delta}_2\pi^{2BB}(1,1;\mu) + (1-\underline{\delta}_2)\pi^{2GB}(1,1;\mu) - \pi^{2B}(0,1;\mu),\end{aligned}$$

then $\bar{\delta}_2(1,G) \geq \bar{\delta}_2(1,B))$ and $\underline{\delta}_2(1,G) \leq \underline{\delta}_2(1,B)$.

ii) If $\pi^{2\theta}(1,1;\mu)$ is supermodular in θ and

$$\begin{aligned}&\underline{\delta}_2\pi^{2BG}(1,1;\mu) + (1-\underline{\delta}_2)\pi^{2GG}(1,1;\mu) - \pi^{2G}(0,1;\mu) \leq \\ &\underline{\delta}_2\pi^{2BB}(1,1;\mu) + (1-\underline{\delta}_2)\pi^{2GB}(1,1;\mu) - \pi^{2B}(0,1;\mu),\end{aligned}$$

then $\bar{\delta}_2(1,G) < \bar{\delta}_2(1,B))$ and $\underline{\delta}_2(1,G) \geq \underline{\delta}_2(1,B)$.

iii) If

$$\begin{aligned}&\underline{\delta}_2\pi^{2BG}(1,1;\mu) + (1-\underline{\delta}_2)\pi^{2GG}(1,1;\mu) - \pi^{2G}(0,1;\mu) \\ \leq &\underline{\delta}_2\pi^{2B}(1,0;\mu) + (1-\underline{\delta}_2)\pi^{2G}(1,0;\mu) - \pi^{2}(0,0;\mu) \\ \leq &\underline{\delta}_2\pi^{2BB}(1,1;\mu) + (1-\underline{\delta}_2)\pi^{2GB}(1,1;\mu) - \pi^{2B}(0,1;\mu),\end{aligned}$$

then $\mathbb{P}_\delta(d_2(1,G)=1) \leq \mathbb{P}_\delta(d_2(0,\theta_1)=1) \leq \mathbb{P}_\delta(d_2(1,B)=1)$ for $\theta_1 \in \{B,G\}$.

When the leader and the follower implement their innovations, i.e., $d = (1,1)$, and the follower's profits are submodular in the state θ and the incremental value of innovation is larger when the leader's state is good, the follower is more likely to implement its innovation and conducts, on average, more R&D when the leader's innovation is good than when it is bad. This happens because the competition intensity faced by the follower is lower when the leader's innovation succeeds than when it fails. When profits are supermodular in θ and the incremental value ranking is reversed, the opposite happens.

The last part establishes conditions for the follower's probability of innovation to be larger when the leader innovates and its state is bad than when the leader does not innovate, and this is larger than when it innovates successfully. The conditions are: i) the follower's marginal profits across states from innovation when the leader innovates and its state is good fall more than the profits when it does not innovate and these are lower when the leader's state is bad; and ii) when the marginal decrease in the incremental value with $\underline{\delta}_2$ is smaller when the leader innovates and its state is good than when the leader does not innovate, and this lower when the leader innovate and its state is bad. Thus, the leader's innovation has a preemptive effect when its idea is good and an encouraging effect when it is bad. This last effect is not present in winner-takes-all markets, as a successful innovation renders competitors' potential innovations useless. Additionally, this effect is not considered in models that do not account for the possibility that not only the innovation but also the realized uncertainty determines whether the firm is leading, lagging, or neck-to-neck.

Firm 1's expected profits are

$$
\begin{aligned}
&\mathbb{E}_{\tau_2,d_2,\delta_2,\delta_1}\left[\int_{\tau_1+\tau_2(d_1,\theta_1)}^{\infty} e^{-rt}\left(d_1\,\pi^{1\theta}(1,d_2(2,\theta_1));\mu) + (1-d_1)\,\pi^1(0,d_2(0,\theta_1);\mu)\right)dt\right] + \\
&\mathbb{E}_{\tau_b,d_2,\delta_2,\delta_1}\left[\int_{\tau_1}^{\tau_1+\tau_2(d_1,\theta_1)} e^{-rt}\left(d_1\pi^{1\theta_1}(1,0;\mu) + (1-d_1)\,\pi^1(0,0;\mu)\right)dt\right] + \\
&\int_0^{\tau_1} e^{-rt}\big(\pi^1(0,0;\mu) - c\big)dt.
\end{aligned}
$$

From the initial time to the time the leader stops R&D, it gets its regular profits minus the cost of R&D. After that, if the leader implements its idea, it gets the benefit of its innovation while the follower is producing with the original technology, which occurs between τ_1 and $\tau_1 + \tau_2$. The leader gets the regular profits when it does not implement its idea. Finally, after the follower stops conducting R&D, the leader obtains the expected profits when d_1 is chosen and the follower chooses d_2. When the follower implements its idea and it is good, the leader's expected profits drop; conversely, when the follower's innovation turns out to be bad, the leader's profits increase.

Let's define the following function

$$R(\delta_1) = \max\Big\{\mathbb{E}_{\tau_2,d_2,\delta_2}\Big[\delta_1 e^{-r\,\tau_2(1,B)}\big(\pi^{1B\theta_2}(1,d_2(1,B);\mu)-\pi^{1B}(1,0;\mu)\big)+\pi^{1B}(1,0;\mu)\big)+$$
$$(1-\delta_1)e^{-r\,\tau_2(1,G)}\big(\pi^{1G\theta_2}(1,d_2(1,G);\mu)-\pi^{1G}(1,0;\mu)\big)+\pi^{1G}(1,0;\mu)\big)-\pi^{1}(0,0;\mu)+c\Big],$$
$$\mathbb{E}_{\tau_2,d_2,\delta_2}\big[e^{-r\,\tau_2(0,G)}\big(\pi^{1}(0,d_2(0,G);\mu)-\pi^{1}(0,0;\mu)\big)\big]+c\Big\}.$$

Firm 1's optimal stopping problem at time zero is given by

$$\mathcal{R}(\delta_1) = \sup_{\tau_1\in\mathbb{T}}\left\{\mathbb{E}_{\delta_1}\left[e^{-r\tau_1}R(\delta_{1\tau_1})\right]\right\} \tag{25}$$

subject to

$$\mathrm{d}\delta_{1t} = \delta_{1t}\,(1-\delta_{1t})\,\sigma\,\mathrm{d}B_t \quad \text{and} \quad \delta_{10}=\delta_1.$$

In the intervention region, the third QVI condition implies that $\mathcal{R}(\delta)$ solves $(\mathcal{H}\mathcal{R})(\delta_1)=0$, that is,

$$\frac{(\sigma\,\delta_1\,(1-\delta_1))^2}{2}\,\mathcal{R}''(\delta_1) - r\,\mathcal{R}(\delta_1) = 0. \tag{26}$$

The two independent solutions to this ODE are given by $F(\delta_1)$ and $F(1-\delta_1)$ with

$$F(\delta_1) \equiv \frac{(1-\delta_1)^{\gamma}}{\delta_1{}^{\gamma-1}} \quad \text{where } \gamma \equiv \frac{1+\sqrt{1+8\,r/\sigma^2}}{2} \tag{27}$$

and the general solution to (26) is

$$\mathcal{R}(\delta_1) = A_0\,F(\delta_1) + A_1\,F(1-\delta_1), \tag{28}$$

where A_0 and A_1 are the constants of integration, whose values are determined by imposing value-matching $\mathcal{R}(\delta_1) = 1$ and smooth-pasting condition $\mathcal{R}_{\delta}(\delta_1) = 0$ at $\delta_1 = \bar{\delta}_1$. The smooth-pasting condition being equal to zero follows from the fact that the payoff from discarding the innovation is independent of θ_1 since the follower's utility is independent of the realized state for the leader's innovation when the leader does not innovate; that is,

$$\mathbb{E}_{\tau_2,d_2,\delta_2}\big[e^{-r\,\tau_2(0,G)}\big(\pi^{1}(0,d_2(0,G);\mu)-\pi^{1}(0,0;\mu)\big)\big] = \mathbb{E}_{\tau_2,d_2,\delta_2}\big[e^{-r\,\tau_2(0,B)}\big(\pi^{1}(0,d_2(0,B);\mu)-\pi^{1}(0,0;\mu)\big)\big].$$

This gives rise to the following auxiliary solution

$$\widehat{\mathcal{R}}(\delta_1;\bar{\delta}_1) \equiv \begin{cases} (\gamma-\bar{\delta}_1)\,\frac{F(\delta_1)}{F(\bar{\delta}_1)} + (\gamma+\bar{\delta}_1-1)\,\frac{F(1-\delta_1)}{F(1-\bar{\delta}_1)} & \text{if} \quad 0 \le \delta < \bar{\delta}_1, \\ 1 & \text{if} \quad \bar{\delta}_!^\star \le \delta \le 1. \end{cases} \tag{29}$$

This is decreasing and strictly convex in $\delta_1 \in (0,1)$ for a fixed $\bar{\delta}_1$, which are properties that we use to derive the next result. Also, $\widehat{\mathcal{R}}(\delta_1;\bar{\delta}_1)$ increases with $\bar{\delta}_1$ for any given δ_1.

The following result follows from the previous discussion and Proposition 1.

Proposition 9. *Suppose Assumption 2 holds. Then, firm 1's expected profit is given by* $\frac{1}{r}(\pi^1(0,0;\mu) - c + \mathcal{R}(\delta;\eta))$, *where*[8]

$$\mathcal{R}(\delta_1;\delta_2,\rho) = \begin{cases} \mathbb{E}_{\tau_2,d_2,\delta_2,\delta_1}\big[e^{-r\,\tau_2(1,\theta_1)}\big(\pi^{1\theta}(1,d_2(1,\theta_1);\mu) - \pi^{1\theta_1}(1,0;\mu)\big)\big] + \\ \pi^{1\theta_1}(1,0;\mu) - \pi^1(0,0;\mu) + c & \text{if} \quad 0 \le \delta \le \underline{\delta}_1^\star, \\ \widehat{\mathcal{R}}(\delta_1;\bar{\delta}_1)\big(\mathbb{E}_{\tau_2,d_2,\delta_2}\big[e^{-r\,\tau_2(0,G)}\big(\pi^{1\theta_2}(0,d_2(0,G);\mu) - \pi^1(0,0;\mu)\big)\big] + c\big) & \text{if} \quad \underline{\delta}_1^\star < \delta < \bar{\delta}_1^\star, \\ \mathbb{E}_{\tau_2,d_2,\delta_2}\big[e^{-r\,\tau_2(0,G)}\big(\pi^{1\theta_2}(0,d_2(0,G);\mu) - \pi^1(0,0;\mu)\big)\big] + c & \text{if} \quad \bar{\delta}_1^\star \le \delta \le 1, \end{cases} \tag{30}$$

and the thresholds $\underline{\delta}_1^\star$ *and* $\bar{\delta}_1^\star$ *are determined imposing value-matching and smooth-pasting conditions at* $\delta = \underline{\delta}_1^\star$ *and* $\delta = \bar{\delta}_1^\star$. *The profit-maximizing strategy* $(\tau_1^\star, d_1^\star)$ *is given by*

$$\tau_1^\star = \inf\big\{t > 0\colon \delta_{1t} \notin (\underline{\delta}_1^\star, \bar{\delta}_1^\star)\big\} \qquad \textit{and} \qquad d_1^\star = \mathbb{I}(\delta_{1\tau_1^\star} \le \underline{\delta}_1^\star).$$

Thus, facing competition by a follower does not change the structure of the leader's R&D strategy. It is still optimal to use a two-cutoff strategy since the profits are still quasi-linear, the updating is a martingale, and the payoffs in each state are independent of the time spent doing R&D. However, both the payoff from discarding and implementing the idea are different and depend on what the follower will do. If the follower does not implement its innovation, payoffs are the same as when the leader faces no competition. In contrast, if the follower implements its innovation with positive probability, the leader's payoff from both implementing and discarding its innovation decreases. However, a closed-form solution is unavailable, making it challenging to compare firm 1's optimal R&D strategy across different scenarios.

[8] We could replace $\mathbb{E}_{\tau_b,\delta_2}\big[e^{-r\,\tau_2(0,G)}\big(\pi^{1\theta_2}(0,d_2;\mu) - \pi^i(0,0;\mu)\big) + c\big]$ by $\mathbb{E}_{\tau_2,\delta_2}\big[e^{-r\,\tau_2(0,B)}\big(\pi^{1\theta_2}(0,d_2;\mu) - \pi^i(0,0;\mu)\big) + c\big]$ without modifying the result because their values are the same since when firm 1 does not implement its innovation, which firm 1's state realizes is irrelevant.

The value-matching and smooth pasting conditions that determine $(\underline{\delta}_1, \bar{\delta}_1)$ are given by

$$
\begin{aligned}
&\widehat{\mathcal{R}}(\delta_1;\bar{\delta}_1)\big(\mathbb{E}_{\tau_2,d_2,\delta_2}\big[e^{-r\,\tau_2(0,G)}\big(\pi^{1\theta_2}(0,d_2(0,G);\mu)-\pi^1(0,0;\mu)\big)\big]+c\big)\big|_{\delta_1=\underline{\delta}} = \\
&\Big(\mathbb{E}_{\tau_2,d_2,\delta_2,\delta_1}\big[e^{-r\,\tau_2(1,\theta_1)}\big(\pi^{1\theta}(1,d_2(1,\theta_1);\mu)-\pi^{1\theta_1}(1,0;\mu)\big)\big]+ \\
&\delta_1\pi^{1B}(1,0;\mu)+(1-\delta_1)\pi^{1G}(1,0;\mu)-\pi^1(0,0;\mu)+c\Big)\big|_{\delta_1=\underline{\delta}}
\end{aligned}
\tag{31}
$$

and

$$
\begin{aligned}
&\widehat{\mathcal{R}}_\delta(\delta_1;\bar{\delta}_1)\big(\mathbb{E}_{\tau_2,d_2,\delta_2}\big[e^{-r\,\tau_2(0,G)}\big(\pi^{1\theta_2}(0,d_2(0,G);\mu)-\pi^1(0,0;\mu)\big)\big]+c\big)\big|_{\delta_1=\underline{\delta}} = \\
&\Big(\mathbb{E}_{\tau_2,d_2,\delta_2}\big[e^{-r\,\tau_2(1,B)}\big(\pi^{1B\theta_2}(1,d_2(1,B);\mu)-\pi^{1B}(1,0;\mu)\big)\big]+\pi^{1B}(1,0;\mu)- \\
&\mathbb{E}_{\tau_2,d_2,\delta_2}\big[e^{-r\,\tau_2(1,G)}\big(\pi^{1G\theta_2}(1,d_2(1,G);\mu)-\pi^{1G}(1,0;\mu)\big)\big]-\pi^{1G}(1,0;\mu)\Big)\big|_{\delta_1=\underline{\delta}}.
\end{aligned}
\tag{32}
$$

To understand the impact of competition is illustrative to write these equations as follows

$$
\begin{aligned}
\widehat{\mathcal{R}}(\delta_1;\bar{\delta}_1) =& \frac{\mathbb{E}_{\tau_2,d_2,\delta_2}\big[\text{Incremental Value of Innovation}\big]}{\mathbb{E}_{\tau_2,d_2,\delta_2}\big[\text{Incremental Value of Regular Profits}\big]} \\
\widehat{\mathcal{R}}_\delta(\delta_1;\bar{\delta}_1) =& \frac{\mathbb{E}_{\tau_2,d_2,\delta_2}\big[\text{Change in Incremental Value with }\delta_1\big]}{\mathbb{E}_{\tau_2,d_2,\delta_2}\big[\text{Incremental Value of Regular Profits}\big]}.
\end{aligned}
$$

The right-hand side of the value-matching and smooth-pasting conditions has all the relevant information to know how the leader's R&D strategy changes because of the presence of an active follower. The follower's implementation decision not only impacts the incremental value of innovation and its marginal value change with δ_1, but also regular profits, referred to as the incremental value of regular profits.

Arrow (1962) argues that a leader has less incentive to invest in R&D than a follower, as a leader's innovation cannibalizes its existing rents. When the follower implements its innovation and its state is good, the value cannibalized is smaller than that when the follower does not innovate. The opposite happens when the follower's realized state is bad. Holding the incremental value of innovation and its change constant, this means that the leader has an incentive to choose a higher $\bar{\delta}$ and a lower $\underline{\delta}$ whne the state is good and the opposite when the state is bad.

When the follower implements its innovation with positive probability, the leader's incremental value and its marginal change increase when the follower's state is bad, and decrease when it is good. Thus, the total effect depends on the prior belief that the follower's project is good. In contrast to Gilbert and Newbery (1982), who show that the leader has more incentive to invest in R&D than a follower, since the leader's profit loss of sharing the market is larger than the follower's gain.[9] In our setting, this is not necessarily the case since our innovating leader is not a monopoly; it faces competition from at least $n-2$ lagging firms, and the outcome of innovation depends on the realized state in such a way that when the bad is realized, it is prejudicial. Thus, in our case, the leader could have less incentive to innovate. This is because the leader's profit loss from sharing the market with a successful follower could be either smaller than the follower's gain.

When the ratio on the RHS of the value-matching condition rises with the follower's probability of implementing its innovation and the RHS of the smooth-pasting condition falls with it, competition by the follower increases the probability that the leader implements its innovation and shortens the expected time conducting R&D. The opposite happens when RHS of the value-matching conditions falls and that of the smooth-pasting condition rises with the follower's probability of implementing its innovation.

Let the probability that the leader implements its innovation when facing competition by a follower be $\mathbb{P}_{\delta_1}(d_1 = 1|\delta_2)$. We deduce the following result from Propositions 3 and 4 and by comparing the value-matching and smooth-pasting conditions for the solo innovator with those when there is an active follower.

Proposition 10. *Suppose that $\delta_1 \in (\underline{\delta}_1^\star, \bar{\delta}_1^\star)$. If*

$$\begin{aligned}&\mathbb{E}_{\delta_1,\delta_2}\big[e^{-r\,\tau_2(1,\theta_1)}\big]P(d_2(1,\theta_1)=1)\big(\pi^{1\theta}(1,1;\mu)-\pi^{1\theta_1}(1,0;\mu)\big) \geq\\ &1+c^{-1}\mathbb{E}_{\tau_2,\delta_2}\big[e^{-r\,\tau_2(0,G)}\big]P(d_2(0,G)=1)\big(\pi^{1\theta_2}(0,1;\mu)-\pi^{1}(0,0;\mu)\big)\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}_{\delta_2}\big[e^{-r\,\tau_2(1,B)}\big]P(d_2(1,B)=1)\big(\pi^{1B\theta_2}(1,1;\mu)-\pi^{1B}(1,0;\mu)\big) \leq\\ &\mathbb{E}_{\delta_2}\big[e^{-r\,\tau_2(1,G)}\big]P(d_2(1,G)=1)\big(\pi^{1G\theta_2}(1,1;\mu)-\pi^{1G}(1,0;\mu)\big),\end{aligned}$$

Then, $\mathbb{P}_{\delta_1}(d_1 = 1|\delta_2) > \mathbb{P}_{\delta_1}(d_1 = 1)$. Otherwise, $\mathbb{P}_{\delta_1}(d_1 = 1|\delta_2) < \mathbb{P}_{\delta_1}(d_1 = 1)$.

When the leader faces a competitor that innovates with positive probability, the leader's profits in each state fall. However, the impact of having a competitor depends on the impact of its innovation on the leader's regular profits, i.e., the expected profits the leader would receive if it did not implement its innovation, and the impact on the leader's incremental value of innovation.

[9] Because monopoly profits are usually larger than twice the duopolistic profits ($\pi^m > 2\pi^d$), the leader's profit loss from becoming a duopoly $\pi^m - \pi^d$ is larger than the follower's value of innovating π^d.

The first condition ensures that the incremental value of innovation is larger when the leader faces a follower than when it is the solo innovator. The second establishes that the profit gain from transitioning from a good state to a bad state when facing a follower is larger than when it does not face one. Thus, the leader benefits more from innovation when facing a follower than when it is the sole innovator.

Thus, when the leader's incremental value when the follower implements its innovation is larger than when the follower does not implement it, the leader implements its innovation more often and conducts, on average, less R&D. Otherwise, the opposite happens. Hence, the relationship between competition and R&D depends on the impact that the follower's innovation has on the leader's incremental value of innovation and its marginal change, and on its regular profits, and the effect that the leader's innovation has on the follower's decision to implement its innovation.

Concerning the empirical evidence about the underlying mechanism studied here, Igami (2017) studies the relationship between competition and innovation by focusing on the propensity to innovate of new entrants relative to incumbents in the hard drive industry. He finds that despite strong preemptive motives and a substantial cost advantage over entrants, cannibalization makes incumbents reluctant to innovate, which can explain at least 57% of the incumbent-entrant innovation gap. Hence, the replacement effect appears to be stronger than the preemption effect. Igami and Uetake (2019) study a stochastically alternating-move game of dynamic oligopoly and estimate it using data from the hard disk drive industry, in which a dozen global players consolidated into only three in the last 20 years. They find plateau-shaped equilibrium relationships between competition and innovation, with heterogeneity across time and productivity.

Thus, the evidence, although scarce, confirms that the impact of competition on R&D is industry-dependent, and that both the incremental value of an innovation and the incremental value of regular profits (the impact on the replacement effect) are key, varying depending on the industry.

6 Concluding Remarks

This paper develops a continuous-time model of R&D in which innovation is modeled as a sequential Bayesian learning process. Firms acquire information about the profitability of uncertain innovations over time and decide optimally when to implement, abandon, or continue experimentation.

Our framework departs from traditional patent race and winner-take-all models by considering environments in which innovations are non-drastic, incremental, and non-exclusive. We study the innovation behavior of solo innovators with one or two sequential and correlated ideas, as well as strategic interactions between leader and follower firms.

The theoretical results show that the relationship between competition and innovation is context-dependent. In particular, we find that: (i) an increase in competition intensity generally leads to longer experimentation periods, as measured by the expected time conducting R&D; (ii) however, the probability that the innovation is implemented may rise or fall with competition, depending on how competition affects the *incremental value of innovation*, i.e., the difference between expected profits from innovation and profits from current technology, as well as its change with the belief that the idea is bad; (iii) when firms explore sequential, correlated ideas, stronger correlation improves experimentation efficiency and increases the likelihood of implementation if the first idea has a higher learning speed; and (iv) when firms compete sequentially, the presence of a follower can induce either a *preemptive* or *encouraging* effect on the leader's R&D strategy. This depends on the impact of innovation on the incremental value, its change, and on the impact on the replacement effect.

These findings provide a richer understanding of innovation dynamics in markets where technological rivalry is ongoing, information arrives gradually, and innovations do not fully displace one another. The framework is consistent with several empirical settings, including pharmaceutical R&D, microprocessors, consumer electronics, and battery technology development, where innovation is cumulative, uncertain, and shaped by market structure.

Our model offers implications for innovation policy. First, policies aimed at encouraging innovation should recognize that increased competition does not universally foster R&D. In settings where competition reduces the incremental value of innovation—by compressing post-innovation profits more than pre-innovation profits—firms may delay implementation or abandon innovation altogether. In such cases, targeted R&D subsidies, cost-sharing mechanisms, or temporary IP protections could offset these disincentives in the event that social efficiency demands so.

Second, regulatory interventions that enhance transparency and information spillovers (e.g., open science frameworks or clinical trial registries) may complement innovation by increasing learning speed, especially when firms face correlated ideas.

Third, policies that affect market entry or structure should account for dynamic interactions between incumbents and entrants. For example, if a follower's innovation discourages an incumbent from innovating, strengthening forward-looking IP mechanisms or licensing rules may help maintain innovation incentives without relying on exclusivity.

Lastly, fostering a competitive but not overly fragmented market structure may be optimal. This aligns with empirical findings (e.g., Aghion et al. (2005)) of an inverted-U relationship between competition and innovation, which our model reproduces under specific parameter configurations.

Overall, this paper contributes a tractable and generalizable framework that highlights the nuanced interplay between belief formation, competition, and innovation. Future work may explore richer dynamics, including endogenous entry, leapfrogging innovations, or the role of data-driven feedback in real-time learning environments.

References

Philippe Aghion, Christopher Harris, Peter Howitt, and John Vickers. Competition, imitation and growth with step-by-step innovation. *Review of Economic Studies*, 68(3):467–492, 2001. URL https://EconPapers.repec.org/RePEc:oup:restud:v:68:y:2001:i:3:p:467-492. 2

Philippe Aghion, Nick Bloom, Richard Blundell, Rachel Griffith, and Peter Howitt. Competition and Innovation: an Inverted-U Relationship. *The Quarterly Journal of Economics*, 120(2):701–728, 2005. URL https://ideas.repec.org/a/oup/qjecon/v120y2005i2p701-728..html. 2, 4, 19, 35

Rabah Amir. Cournot oligopoly and the theory of supermodular games. *Games and Economic Behavior*, 15(2):132–148, 1996. URL https://EconPapers.repec.org/RePEc:eee:gamebe:v:15:y:1996:i:2:p:132-148. 6

V.F. Araman and R.A. Caldentey. Diffusion approximations for a class of sequential experimentation problems. *Management Science*, 68(8):5958–5979, 2022. 11

Kenneth Arrow. Economic welfare and the allocation of resources for invention. In *The Rate and Direction of Inventive Activity: Economic and Social Factors*, pages 609–626. National Bureau of Economic Research, Inc, 1962. URL https://EconPapers.repec.org/RePEc:nbr:nberch:2144. 2, 4, 16, 19, 31

Susan Athey and Armin Schmutzler. Investment and Market Dominance. *RAND Journal of Economics*, 32(1):1–26, Spring 2001. URL https://ideas.repec.org/a/rje/randje/v32y2001i1p1-26.html. 6

Martin Neil Baily, Hans Gersbach, FM Scherer, and Frank R Lichtenberg. Efficiency in manufacturing and the need for global competition. *Brookings Papers on Economic Activity. Microeconomics*, 1995: 307–358, 1995. 19

Felipe Balmaceda. Private vs. public communication: Difference of opinion and reputational concerns. *R&R Journal of Economic Theory*, 2020. 6

Felipe Balmaceda, René Caldentey, and Maia Cufré. Information acquisition and execution in projects with uncertain returns. Instituto de Política Económica, Andrés Bello University; Booth School of Business, The University of Chicago; Northwestern University, July 2025. 3

Pilar Beneito, Paz Coscollá-Girona, María Engracia Rochina-Barrachina, and Amparo Sanchis. Competitive pressure and innovation at the firm level. *The Journal of Industrial Economics*, 63(3):422–457, 2015. doi: 10.1111/joie.12079. URL https://onlinelibrary.wiley.com/doi/abs/10.1111/joie.12079. 20

Nicholas Bloom, Mark Schankerman, and John Van Reenen. Identifying technology spillovers and product market rivalry. *Econometrica*, 81(4):1347–1393, 2013. ISSN 1468-0262. doi: 10.3982/ECTA9466. URL http://dx.doi.org/10.3982/ECTA9466. 25

Richard Blundell, Rachel Griffith, and John van Reenen. Dynamic count data models of technological innovation. *Economic Journal*, 105(429):333–44, 1995. URL https://EconPapers.repec.org/RePEc:ecj:econjl:v:105:y:1995:i:429:p:333-44. 19

Jan Boone. Competitive Pressure: The Effects on Investments in Product and Process Innovation. *RAND Journal of Economics*, 31(3):549–569, Autumn 2000. URL https://ideas.repec.org/a/rje/randje/v31y2000iautumnp549-569.html. 4, 6

Jan Boone. Intensity of competition and the incentive to innovate. *International Journal of Industrial Organization*, 19(5):705–726, April 2001. URL https://ideas.repec.org/a/eee/indorg/v19y2001i5p705-726.html. 4, 6

Fernando Branco, Monic Sun, and J Miguel Villas-Boas. Optimal search for product information. *Management Science*, 58(11):2037–2056, 2012. 3

H. Chernoff. Sequential design of experiments. *Ann. Math. Statist.*, 30(3):755–770, 09 1959. doi: 10.1214/aoms/1177706205. 3

H. Chernoff. *Sequential Analysis and Optimal Design*. SIAM, Philadelphia, PA, 1972. 3

Claude d'Aspremont and Alexis Jacquemin. Cooperative and noncooperative r&d in duopoly with spillovers. *American Economic Review*, 78(5):1133–37, 1988. URL https://EconPapers.repec.org/RePEc:aea:aecrev:v:78:y:1988:i:5:p:1133-37. 25

Federico Etro. Innovation by leaders. *Economic Journal*, 114(495):281–303, April 2004. URL https://ideas.repec.org/a/ecj/econjl/v114y2004i495p281-303.html. 4

Drew Fudenberg, Philipp Strack, and Tomasz Strzalecki. Speed, accuracy, and the optimal timing of choices. *American Economic Review*, 108(12):3651–3684, 2018. 3

Richard Gilbert, Christian Riis, and Erlend S. Riis. Stepwise innovation by an oligopoly. *International Journal of Industrial Organization*, 61:413 – 438, 2018. ISSN 0167-7187. doi: https://doi.org/10.1016/j.ijindorg.2018.10.001. URL http://www.sciencedirect.com/science/article/pii/S0167718718300997. 4

Richard J Gilbert and David M G Newbery. Preemptive Patenting and the Persistence of Monopoly. *American Economic Review*, 72(3):514–526, June 1982. URL https://ideas.repec.org/a/aea/aecrev/v72y1982i3p514-26.html. 2, 4, 31

Ronald L. Goettler and Brett R. Gordon. Does amd spur intel to innovate more? *Journal of Political Economy*, 119(6):1141–1200, 2011. doi: 10.1086/664615. URL https://doi.org/10.1086/664615. 20

Aamir Rafique Hashmi. Competition and Innovation: The Inverted-U Relationship Revisited. *The Review of Economics and Statistics*, 95(5):1653–1668, 12 2013. ISSN 0034-6535. doi: 10.1162/REST_a_00364. URL https://doi.org/10.1162/REST_a_00364. 20

Aamir Rafique Hashmi and Johannes Van Biesebroeck. The Relationship between Market Structure and Innovation in Industry Equilibrium: A Case Study of the Global Automobile Industry. *The Review of Economics and Statistics*, 98(1):192–208, 03 2016. ISSN 0034-6535. doi: 10.1162/REST_a_00494. URL https://doi.org/10.1162/REST_a_00494. 20

Mitsuru Igami. Estimating the innovators dilemma: Structural analysis of creative destruction in the hard disk drive industry, 19811998. *Journal of Political Economy*, 125(3):798–847, 2017. doi: 10.1086/691524. URL https://doi.org/10.1086/691524. 33

Mitsuru Igami and Kosuke Uetake. Mergers, Innovation, and Entry-Exit Dynamics: Consolidation of the Hard Disk Drive Industry, 19962016. *The Review of Economic Studies*, 87(6):2672–2702, 09 2019. ISSN 0034-6527. doi: 10.1093/restud/rdz044. URL https://doi.org/10.1093/restud/rdz044. 33

Tobias Kretschmer, Eugenio J. Miravete, and Jose C. Pernias. Competitive Pressure and the Adoption of Complementary Innovations. *American Economic Review*, 102(4):1540–1570, June 2012. URL https://ideas.repec.org/a/aea/aecrev/v102y2012i4p1540-70.html. 20

Ruitian Lang. Try before you buy: A theory of dynamic information acquisition. *Journal of Economic Theory*, 183:1057–1093, 2019. 3

Tome Lee and Louis L. Wilde. Market Structure and Innovation: A Reformulation. *The Quarterly Journal of Economics*, 94(2):429–436, 1980. URL https://ideas.repec.org/a/oup/qjecon/v94y1980i2p429-436..html. 4

Igor Letina. The road not taken: competition and the R&D portfolio. *RAND Journal of Economics*, 47(2): 433–460, May 2016. URL https://ideas.repec.org/a/bla/randje/v47y2016i2p433-460.html. 4

Ángel L. López and Xavier Vives. Overlapping Ownership, R&D Spillovers, and Antitrust Policy. Technical report, 2016. 4, 25

David A. Malueg and Shunichi O. Tsutsui. Dynamic R&D Competition with Learning. *RAND Journal of Economics*, 28(4):751–772, Winter 1997. URL https://ideas.repec.org/a/rje/randje/v28y1997iwinterp751-772.html. 4

Guillermo Marshall and Álvaro Parra. Innovation and competition: The role of the product market. *International Journal of Industrial Organization*, 65:221 – 247, 2019. ISSN 0167-7187. doi: https://doi.org/10.1016/j.ijindorg.2019.04.001. URL http://www.sciencedirect.com/science/article/pii/S0167718719300207. 4

Paul Milgrom and John Roberts. Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica*, 58(6):1255–77, November 1990. URL http://ideas.repec.org/a/ecm/emetrp/v58y1990i6p1255-77.html. 6

Giuseppe Moscarini and Lones Smith. The optimal level of experimentation. *Econometrica*, 69(6): 1629–1644, 2001. 3

Stephen Nickell. Competition and corporate performance. *Journal of Political Economy*, 104(4):724–46, 1996. URL https://EconPapers.repec.org/RePEc:ucp:jpolec:v:104:y:1996:i:4:p:724-46. 19

Bernt Øksendal. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013. 14, 45

Alvaro Parra. Sequential innovation, patent policy, and the dynamics of the replacement effect. *RAND Journal of Economics*, 50(3):568–590, September 2019. doi: 10.1111/1756-2171.12287. URL https://ideas.repec.org/a/bla/randje/v50y2019i3p568-590.html. 4

Jeffrey M. Perloff and Steven C. Salop. Equilibrium with Product Differentiation. *Review of Economic Studies*, 52(1):107–120, 1985. URL https://ideas.repec.org/a/oup/restud/v52y1985i1p107-120..html. 6

Goran Peskir and Albert Shiryaev. *Optimal stopping and free-boundary problems*. Springer, 2006. 7

Jennifer F Reinganum. A Dynamic Game of R and D: Patent Protection and Competitive Behavior. *Econometrica*, 50(3):671–688, May 1982. URL https://ideas.repec.org/a/ecm/emetrp/v50y1982i3p671-88.html. 4

Jennifer F Reinganum. Uncertain Innovation and the Persistence of Monopoly. *American Economic Review*, 73(4):741–748, September 1983. URL https://ideas.repec.org/a/aea/aecrev/v73y1983i4p741-48.html. 2, 4

Kevin Roberts and Martin L Weitzman. Funding criteria for research, development, and exploration projects. *Econometrica: Journal of the Econometric Society*, pages 1261–1288, 1981. 3

Armin Schmutzler. Competition and investment A unified approach. *International Journal of Industrial Organization*, 31(5):477–487, 2013. doi: 10.1016/j.ijindorg.2013.0. URL https://ideas.repec.org/a/eee/indorg/v31y2013i5p477-487.html. 4, 6

J.A. Schumpeter. *Capitalism, Socialism and Democracy*. Harper & Brothers, 1942. ISBN 9781134841509. URL https://books.google.cl/books?id=ytrqJswoRCoC. 2

Carl Shapiro. Competition and innovation: Did arrow hit the bull's eye? In *The Rate and Direction of Inventive Activity Revisited*. University of Chicago Press, 04 2012. ISBN 9780226473031. doi: 10.7208/chicago/9780226473062.003.0011. URL https://doi.org/10.7208/chicago/9780226473062.003.0011. 2

D. Siegmund. *Sequential Analysis: Tests and Confidence Intervals.* Springer-Verlag, New York, NY, 1985. 3

Canan Ulu and James E. Smith. Uncertainty, information acquisition, and technology adoption. *Operations Research*, 57(3):740–752, 2009. 3

Xavier Vives. Innovation and competitive pressure. *Journal of Industrial Economics*, 56(3):419–469, December 2008. URL http://ideas.repec.org/a/bla/jindec/v56y2008i3p419-469.html. 4

Abraham Wald. Sequential tests of statistical hypotheses. *Annals of Mathematical Statistics*, 16:117–186, 1945. 2, 3

Abraham Wald and Jacob Wolfowitz. Optimum character of the sequential probability ratio test. *Annals of Mathematical Statistics*, 19:326–339, 1948. 3

A Proofs for the Case of One Innovating Firm with One Idea

PROOF OF LEMMA 1:

(a). If $\min\{\pi^{iG}(1;\mu), \pi^{iB}(1;\mu)\} \geq \pi^{i}(0;\mu)$, then it follows from the definition of $V(\delta)$ that for any $\delta \in [0,1]$ $V(\delta) = (\pi^{i\theta}(1;\mu) - (\pi(0;\mu) - c))$. As a result, firm i's optimal stopping problem in (4) satisfies

$$
\begin{aligned}
\mathcal{V}(\delta) = \sup_{\tau \in \mathbb{T}} \mathbb{E}_{\delta}\left[e^{-r\tau} V(\delta_\tau)\right] &= \sup_{\tau \in \mathbb{T}} \mathbb{E}_{\delta}\left[e^{-r\tau}\left(\pi^{i\theta}(1;\mu) - (\pi(0;\mu) - c)\right)\right] \\
&\leq \sup_{\tau \in \mathbb{T}} \mathbb{E}_{\delta}\left[\pi^{i\theta}(1;\mu) - (\pi(0;\mu) - c)\right] \\
&= \mathbb{E}_{\delta}[\pi^{i\theta}(1;\mu)] - (\pi(0;\mu) - c),
\end{aligned}
$$

where the last equality follows from the optional stopping theorem and the fact that δ_t is a bounded continuous martingale. Since $\mathbb{E}_{\delta}[\pi^{i\theta}(1;\mu)] - (\pi(0;\mu) - c) \geq 0$, we conclude that it is optimal for the firm to implement its idea immediately, i.e., $\tau^\star = 0$ and $d^\star = 1$.

(b). The proof follows trivially by noticing that the maximum expected payoff that the firm could get by implementing the idea equals $\max\limits_{\delta \in (0,1)} \mathbb{E}_{\delta}[\pi^{i\theta}(1;\mu)] - (\pi(0;\mu) - c) < 0$. $\square$

PROOF OF THEOREM 1: For an $f \in \widehat{\mathcal{C}}^2$ that solves (QVI) we have

$$
\begin{aligned}
e^{-r\tau} f(\delta_\tau) &= f(\delta) + \int_0^\tau e^{-rt}\, \mathcal{H} f(\delta_t)\, \mathrm{d}t + \int_0^\tau e^{-rt} \sigma\, \delta_t\, (1-\delta_t)\, f'(\delta_t)\, \mathrm{d}B_t \\
&\leq f(\delta) + \int_0^\tau e^{-rt}\, \sigma\, \delta_t\, (1-\delta_t)\, f'(\delta_t)\, \mathrm{d}B_t,
\end{aligned}
$$

where the equality follows from integration-by-parts and Itô's lemma and the inequality follows from the fact that $\mathcal{H}f(\delta) \leq 0$ (second QVI condition). Taking expectation and canceling the stochastic integral, we get $\mathbb{E}[e^{-r\tau} f(\delta_\tau)] \leq f(\delta)$. From the first QVI condition it follows that $\mathbb{E}[e^{-r\tau} V(\delta_\tau)] \leq \mathbb{E}[e^{-r\tau} f(\delta_\tau)] \leq f(\delta)$. Taking the supreme over all stopping times $\tau \geq 0$, we conclude that $f(\delta) \geq \mathcal{V}(\delta)$. Finally, all the inequalities above become equalities for the QVI-control associated to f. This follows from Dynkin's formula and the fact that the QVI-control is the first exit time from the continuation region $\mathcal{C}$. $\square$

Proof of Proposition 1: Let $\mathcal{V}(\delta)$ be the function defined in (11). We will show that $\mathcal{V}(\delta)$ satisfies the (QVI) optimality conditions and so by Theorem 1 it is equal to the firm's optimal expected payoff in (4). To this end, note that $\mathcal{V}(\delta) \in \widehat{\mathcal{C}}^2$, which follows from the smooth-pasting and value-matching conditions. Also, by (28) and the fact that $V(\delta)$ is piece-wise linear, we have that

$$(\mathcal{H}\mathcal{V})(\delta) = \begin{cases} -r\,V(\delta) & \text{if} \quad 0 \leq \delta < \underline{\delta}, \\ \\ 0 & \text{if} \quad \underline{\delta} < \delta < \bar{\delta}, \\ \\ -r\,V(\delta) & \text{if} \quad \bar{\delta} < \delta \leq 1. \end{cases}$$

From this, and the definition of $\mathcal{V}(\delta)$, it follows that $(\mathcal{H}\mathcal{V})(\delta) \leq 0$ and $\big(\mathcal{V}(\delta) - V(\delta)\big)\,(\mathcal{H}\mathcal{V})(\delta) = 0$ for all $\delta \in [0,1] \setminus \{\underline{\delta}, \bar{\delta}\}$. Thus, $\mathcal{V}(\delta)$ satisfies the second and third (QVI) condition.

Next, we show the existence and uniqueness of a function $\mathcal{V}(\delta)$ satisfying the condition in the proposition. To simplify the notation let us define an auxiliary family of functions $\{\widehat{\mathcal{V}}(\delta;\bar{\delta})\colon \delta \in (0,1)\}$ parameterized by $\bar{\delta} \in (0,1)$ such that

$$\widehat{\mathcal{V}}(\delta;\bar{\delta}) = \beta_0(\bar{\delta})\,F(\delta) + \beta_1(\bar{\delta})\,F(1-\delta) \quad \text{if} \quad 0 < \delta < \bar{\delta},$$

where the constants $\beta_0(\bar{\delta})$ and $\beta_1(\bar{\delta})$ are chosen so that $\widehat{\mathcal{V}}(\delta,\bar{\delta})$ is continuously differentiable at $\delta = \bar{\delta}$.

To find $\beta_0(\bar{\delta})$ and $\beta_1(\bar{\delta})$, we impose value-matching and smooth-pasting conditions at $\delta = \bar{\delta}$:

$$\beta_0(\bar{\delta})\,F(\bar{\delta}) + \beta_1(\bar{\delta})\,F(1-\bar{\delta}) = 1 \qquad \text{and} \qquad \beta_0(\bar{\delta})\,F\delta(\bar{\delta}) + \beta_1(\bar{\delta})\,F_\delta(1-\bar{\delta}) = 0.$$

Using the fact that $F_\delta(\delta) = F(\delta)\dfrac{(1-\gamma-\delta)}{\delta\,(1-\delta)}$ and $F_\delta(1-\delta) = F(1-\delta)\dfrac{(\gamma-\delta)}{\delta\,(1-\delta)}$, we get that

$$\beta_0(\bar{\delta}) = \frac{(\gamma-\bar{\delta})}{(2\,\gamma-1)\,F(\bar{\delta})} \qquad \text{and} \qquad \beta_1(\bar{\delta}) = \frac{(\gamma+\bar{\delta}-1)}{(2\,\gamma-1)\,F(1-\bar{\delta})}.$$

Since $\gamma > 1$ it follows that $\beta_0(\bar{\delta})$ and $\beta_1(\bar{\delta})$ are both positive for $\bar{\delta} \in (0,1)$. Furthermore, $\beta_0(\bar{\delta}) \uparrow \infty$ as $\bar{\delta} \uparrow 1$ and $\beta_1(\bar{\delta}) \uparrow \infty$ as $\bar{\delta} \downarrow 0$. It follows that

$$\widehat{\mathcal{V}}(\delta;\bar{\delta}) = \begin{cases} \frac{(\gamma-\bar{\delta})}{(2\,\gamma-1)}\,\frac{F(\delta)}{F(\bar{\delta})} + \frac{(\gamma+\bar{\delta}-1)}{(2\,\gamma-1)}\,\frac{F(1-\delta)}{F(1-\bar{\delta})} & \text{if} \quad 0 < \delta < \bar{\delta}, \\ 1 & \text{if} \quad \bar{\delta} \leq \delta \leq 1. \end{cases} \tag{33}$$

By construction the function $\widehat{\mathcal{V}}(\delta;\bar{\delta})$ is continuously differentiable in $(0,1)$. Furthermore, in the region $\delta \in (0,\bar{\delta})$ it is also decreasing and strictly convex. To see this, note that in this region $\widehat{\mathcal{V}}(\delta;\bar{\delta})$ satisfies the differential equation (26) and so

$$\begin{aligned}\frac{(\sigma\delta(1-\delta))^2}{2}\widehat{\mathcal{V}}''(\delta;\bar{\delta}) - r\,\widehat{\mathcal{V}}(\delta;\bar{\delta}) = 0 \quad \Longrightarrow \quad \widehat{\mathcal{V}}''(\delta;\bar{\delta}) =& \frac{2}{(\sigma\delta(1-\delta))^2}\left(r\widehat{\mathcal{V}}(\delta;\bar{\delta})\right)\\ \geq& \frac{2}{(\sigma\delta(1-\delta))^2} > 0.\end{aligned}$$

This proves that it is strictly convex. In addition, from the smooth-pasting condition $\widehat{\mathcal{V}}'(\bar{\delta};\bar{\delta}) = 0$. As $\widehat{\mathcal{V}}'(\delta;\bar{\delta})$ increases with δ, we get that $\widehat{\mathcal{V}}'(\delta;\bar{\delta}) < 0$ in the region $\delta \in (0,\bar{\delta})$ proving that it is decreasing.

To complete the proof, we will show that there exists a value of

$$\bar{\delta} > \hat{\delta} := (\pi^{iG}(1;\mu) - \pi^{i}(0;\mu))/(\pi^{iG}(1;\mu) - \pi^{iB}(1;\mu)),$$

such that the associated function $\widehat{\mathcal{V}}(\delta;\bar{\delta})$ satisfies value-matching and smooth-pasting conditions with the function $(1-\delta)\,\pi^{iG}(1;\mu) + \delta\,\pi^{iB}(1;\mu)$ at some $\underline{\delta} < \hat{\delta}$.

The argument combines the following facts:

i) The function $\widehat{\mathcal{V}}(\delta;\bar{\delta})$ is monotonically decreasing and strictly convex in $(0,\bar{\delta}]$ as argued above.

ii) The function $V(\delta)$ is piece-wise linear in (0,1).

iii) $\widehat{\mathcal{V}}(\delta;\bar{\delta})$ is monotonic in $\bar{\delta}$, that is, $\widehat{\mathcal{V}}(\delta;\bar{\delta}_1) \leq \widehat{\mathcal{V}}(\delta;\bar{\delta}_2)$ for $\bar{\delta}_1 \leq \bar{\delta}_2$.

iv) For all $\bar{\delta} \in (0,1)$ we have that $\widehat{\mathcal{V}}(\delta;\bar{\delta}) \uparrow \infty$ as $\delta \downarrow 0$.

v) For $\bar{\delta}$ sufficiently large $\widehat{\mathcal{V}}(\delta;\bar{\delta}) > V(\delta)$ for all $\delta \in (0,\bar{\delta})$.

Point (iii) follows from noticing that in (A) the first-factor numerator is null when $\delta = \bar{\delta}$ and decreases with $\bar{\delta}$ as $F'(\delta) < 0$, then the numerator is negative for all $\delta < \bar{\delta}$. The denominator is always positive. Finally, the second factor is negative by $\gamma > 1$.

$$\frac{\partial\widehat{\mathcal{V}}(\delta;\bar{\delta})}{\partial\bar{\delta}} = \frac{1}{2\gamma-1}\left[\frac{F(1-\delta)F(\bar{\delta}) - F(\delta)F(1-\bar{\delta})}{F(1-\bar{\delta})F(\bar{\delta})}\right]\frac{\gamma(1-\gamma)}{\bar{\delta}(1-\bar{\delta})} > 0. \tag{34}$$

Point (iv) follows from noticing that $F(0) \uparrow \infty$ as $\delta \downarrow 0$. Finally, (v) follows the fact that $\beta_0(\bar{\delta})$ grows unboundedly as $\bar{\delta} \uparrow 1$.

Combining points (i) and (iv), it follows that if $\bar{\delta} \le \hat{\delta}$, the function $\widehat{\mathcal{V}}(\delta;\bar{\delta})$ will intersect $V(\delta)$ in a non-smooth way in the region $(0,\bar{\delta})$. Thus, smooth-pasting can only be achieved if $\bar{\delta} > \hat{\delta}$. On the other hand, by point (v) for δ sufficiently large, the function $\widehat{\mathcal{V}}(\delta;\bar{\delta})$ never intersects $V(\delta)$ in $(0,\bar{\delta})$ and so again there is trivially no smooth-pasting in this region. Thus, by the continuity $\widehat{\mathcal{V}}(\delta;\bar{\delta})$ on δ and points (i) and (ii) there exists a $\bar{\delta}$ such that $\widehat{\mathcal{V}}(\delta;\bar{\delta})$ intersects smoothly $V(\delta)$ in the region $(0,\bar{\delta})$. Finally, by point (iii) there is a unique $\bar{\delta} \in (\hat{\delta},1)$ for which $\widehat{\mathcal{V}}(\delta;\bar{\delta})$ satisfies the smooth-pasting condition. □

PROOF OF PROPOSITION 2: To derive the moment generating function $\mathbb{E}_\delta[e^{s\,\tau}]$ of τ, let us consider a function $f(\delta)$ such that $f(\underline{\delta}) = f(\bar{\delta}) = 1$ and

$$\frac{1}{2}\,\sigma^2\,\delta^2\,(1-\delta)^2\,f''(\delta) + s\,f(\delta) = 0 \quad \text{for all } \delta \in [\underline{\delta},\bar{\delta}].$$

For $s < \sigma^2/8$, the solution to this ODE is given by $f(\delta) = K_0\,F(\delta) + K_1\,F(1-\delta)$ for two constants of integration K_0 and K_1, where

$$F(\delta) = \frac{(1-\delta)^{\eta(s)}}{\delta^{\eta(s)-1}} \qquad \text{with } \eta(s) = \frac{1+\sqrt{1-8\,s/\sigma^2}}{2}.$$

We find the values of K_0 and K_1 imposing the boundary conditions $f(\underline{\delta}) = f(\bar{\delta}) = 1$. It follows that

$$f(\delta) = \frac{(F(1-\bar{\delta}) - F(1-\underline{\delta}))\,F(\delta) + (F(\underline{\delta}) - F(\bar{\delta}))\,F(1-\delta)}{F(\underline{\delta})\,F(1-\bar{\delta}) - F(\bar{\delta})\,F(1-\underline{\delta})}.$$

From Dynkin's formula (see Øksendal, 2013), we get

$$\mathbb{E}_\delta[e^{s\tau}\,f(\delta_\tau)] = f(\delta) + \mathbb{E}_\delta\left[\int_0^\tau \left(\frac{1}{2}\,\sigma^2\,\delta^2\,(1-\delta)^2\,f''(\delta) + s\,f(\delta)\right)e^{s\,t}\,\mathrm{d}t\right] = f(\delta).$$

But since $f(\underline{\delta}) = f(\bar{\delta}) = 1$, we have that $\mathbb{E}_\delta[e^{s\tau}\,f(\delta_\tau)] = \mathbb{E}_\delta[e^{s\,\tau}]$. We conclude that

$$\mathbb{E}_\delta[e^{s\,\tau}] = \frac{(F(1-\bar{\delta}) - F(1-\underline{\delta}))\,F(\delta) + (F(\underline{\delta}) - F(\bar{\delta}))\,F(1-\delta)}{F(\underline{\delta})\,F(1-\bar{\delta}) - F(\bar{\delta})\,F(1-\underline{\delta})}.$$

To compute the expected duration of due diligence, $\mathbb{E}_\delta[\tau]$, we can either evaluate the derivative of $\mathbb{E}_\delta[e^{s\,\tau}]$ with respect to s at $s=0$. Alternatively, consider a function $g(\delta)$ such that

$$\frac{1}{2}\,\sigma^2\,\delta^2\,(1-\delta)^2\,g''(\delta) = 1 \quad \text{for all } \delta \in [\underline{\delta},\bar{\delta}].$$

One particular solution is given by

$$g(\delta) = \frac{2\,(1-2\delta)}{\sigma^2}\,\ln\left(\frac{1-\delta}{\delta}\right).$$

Then, it follows that

$$\mathbb{E}_\delta[g(\delta_\tau)] = g(\delta) + \mathbb{E}_\delta\left[\int_0^\tau \frac{1}{2}\,\sigma^2\,\delta^2\,(1-\delta)^2\,g''(\delta)\,\mathrm{d}t\right] = g(\delta) + \mathbb{E}_\delta[\tau].$$

But since $\mathbb{E}_\delta[g(\delta_\tau)] = g(\underline{\delta})\,\mathbb{P}_\delta(\delta_\tau = \underline{\delta}) + g(\bar{\delta})\,\mathbb{P}_\delta(\delta_\tau = \bar{\delta})$, we conclude that

$$\mathbb{E}_\delta[\tau] = \left(\frac{\bar{\delta}-\delta}{\bar{\delta}-\underline{\delta}}\right) g(\underline{\delta}) + \left(\frac{\delta-\underline{\delta}}{\bar{\delta}-\underline{\delta}}\right) g(\bar{\delta}) - g(\delta)$$

.

Finally, we use a similar derivation to compute $\mathbb{P}_\delta(\delta_\tau = \underline{\delta})$ and $\mathbb{P}_\delta(\delta_\tau = \bar{\delta}) = 1 - \mathbb{P}_\delta(\delta_\tau = \underline{\delta})$. Let us define the function $h(\delta)$ such that $h(\underline{\delta}) = 1$, $h(\bar{\delta}) = 0$ and

$$\frac{1}{2}\,\sigma^2\,\delta^2\,(1-\delta)^2\,h''(\delta) = 0 \quad \text{for all } \delta \in [\underline{\delta}, \bar{\delta}].$$

It follows that $h(\delta) = (\bar{\delta}-\delta)/(\bar{\delta}-\underline{\delta})$. Then

$$\mathbb{P}_\delta(\delta_\tau = \underline{\delta}) = \mathbb{E}_\delta[\mathbb{I}(\delta_\tau = \underline{\delta})] = \mathbb{E}_\delta[h(\delta_\tau)] = h(\delta) + \mathbb{E}_\delta\left[\int_0^\tau \frac{1}{2}\,\sigma^2\,\delta^2\,(1-\delta)^2\,h''(\delta)\,\mathrm{d}t\right] = h(\delta) = \frac{\bar{\delta}-\delta}{\bar{\delta}-\underline{\delta}}. \;\square$$

PROOF OF PROPOSITION 3 AND PROPOSITION 4: The function $\widehat{\mathcal{V}}(\delta;\bar{\delta}^\star)$ must satisfy smooth-pasting and value-matching conditions at $\underline{\delta}^\star$, $\bar{\delta}$ satisfying its corresponding smooth-pasting and value-matching conditions.

Thus, value matching and smooth-pasting at $(\underline{\delta}^\star, \bar{\delta}^\star)$ entail the following

$$\widehat{\mathcal{V}}(\underline{\delta}^\star;\bar{\delta}^\star)c = \underline{\delta}^\star \pi^{iB}(1;\mu) + (1-\underline{\delta}^\star)\pi^{iG}(1;\mu) - \pi^i(0;\mu) + c$$

$$\widehat{\mathcal{V}}_\delta(\underline{\delta}^\star;\bar{\delta}^\star)c = \pi^{iB}(1;\mu) - \pi^{iG}(1;\mu)$$

Let $\triangle_\pi(\mu) = \pi^{iB}(1;\mu) - \pi^{iG}(1;\mu)$ and normalize $c = 1$. Totally differentiating both equations with respect to $\bar{\delta}$, $\underline{\delta}$, and $(\pi^{iB}(1;\mu), \pi^{iG}(1;\mu))$, we get that

$$\begin{pmatrix} \widehat{\mathcal{V}}_{\underline{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) - \triangle_\pi(\mu) & \widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) \\ \widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) & \widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) \end{pmatrix} \times \begin{pmatrix} \underline{\delta}^\star_{\pi^G} & \underline{\delta}^\star_{\pi^B} & \underline{\delta}^\star_{\pi} \\ \bar{\delta}^\star_{\pi^G} & \bar{\delta}^\star_{\pi^B} & \bar{\delta}^\star_{\pi} \end{pmatrix} = \begin{pmatrix} 1-\underline{\delta}^\star & \underline{\delta}^\star & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Recall that

$$\widehat{\mathcal{V}}(\delta;\bar{\delta}) = \begin{cases} \frac{(\gamma-\bar{\delta})}{(2\gamma-1)}\,\frac{F(\delta)}{F(\bar{\delta})} + \frac{(\gamma+\bar{\delta}-1)}{(2\gamma-1)}\,\frac{F(1-\delta)}{F(1-\bar{\delta})} & \text{if} \quad 0 < \delta < \bar{\delta} \\ 1 & \text{if} \quad \bar{\delta} \le \delta \le 1. \end{cases} \tag{35}$$

Thus, the determinant of the matrix is given by

$$\begin{aligned}\det \widehat{\mathcal{V}} \equiv & \big((\widehat{\mathcal{V}}_{\underline{\delta}}(\underline{\delta}^{\star};\bar{\delta}^{\star}) - \triangle_{\pi}(\mu))\widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^{\star};\bar{\delta}^{\star}) - \widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta}^{\star};\bar{\delta}^{\star})\widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^{\star};\bar{\delta}^{\star}) = \\ & - \widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta}^{\star};\bar{\delta}^{\star})\widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^{\star};\bar{\delta}^{\star}) < 0\end{aligned}$$

where the inequality readily follows from the fact that smooth-pasting implies that $\widehat{\mathcal{V}}_{\delta}(\underline{\delta}^{\star};\bar{\delta}^{\star}) = \triangle_{\pi}(\mu)$, $\widehat{\mathcal{V}}(\delta;\bar{\delta}^{\star})$ is decreasing and strictly convex in δ and non-decreasing in $\bar{\delta}$ for any given δ, and

$$\frac{\partial \widehat{\mathcal{V}}(\underline{\delta},\bar{\delta})}{\partial \underline{\delta}} = \frac{1}{2\gamma-1}\frac{1}{\underline{\delta}(1-\underline{\delta})}\left((\gamma-\bar{\delta})(1-\gamma-\underline{\delta})\,\frac{F(\delta)}{F(\bar{\delta})} + (\gamma+\bar{\delta}-1)(\gamma-\underline{\delta})\frac{F(1-\delta)}{F(1-\bar{\delta})}\right) < 0,$$

$$\frac{\partial^2 \widehat{\mathcal{V}}(\underline{\delta},\bar{\delta})}{\partial \underline{\delta}\partial \underline{\delta}} = -\frac{\gamma(1-\gamma)}{2\gamma-1}\frac{1}{(\underline{\delta}(1-\underline{\delta}))^2}\left((\gamma-\bar{\delta})\,\frac{F(\delta)}{F(\bar{\delta})} + (\gamma+\bar{\delta}-1)\frac{F(1-\delta)}{F(1-\bar{\delta})}\right) > 0,$$

$$\frac{\partial \widehat{\mathcal{V}}(\delta;\bar{\delta})}{\partial \bar{\delta}} = \frac{1}{2\gamma-1}\left(\frac{F(1-\delta)}{F(1-\bar{\delta})} - \frac{F(\delta)}{F(\bar{\delta})}\right)\frac{\gamma(1-\gamma)}{\bar{\delta}(1-\bar{\delta})} > 0,$$

and

$$\frac{\partial^2 \widehat{\mathcal{V}}(\underline{\delta},\bar{\delta})}{\partial \underline{\delta}\partial \bar{\delta}} = \frac{\gamma(1-\gamma)}{2\gamma-1}\frac{1}{\underline{\delta}(1-\underline{\delta})\bar{\delta}(1-\bar{\delta})}\left((\gamma-\underline{\delta})\,\frac{F(1-\underline{\delta})}{F(1-\bar{\delta})} + (\gamma+\underline{\delta}-1)\frac{F(\underline{\delta})}{F(\bar{\delta})}\right) < 0.$$

Thus,

$$\begin{aligned}\widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta};\bar{\delta}) + \widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta};\bar{\delta}) = & \frac{\gamma(1-\gamma)}{2\gamma-1}\frac{1}{\underline{\delta}(1-\underline{\delta})}\left(\left(\frac{\gamma-\underline{\delta}}{\bar{\delta}(1-\bar{\delta})} - \frac{\gamma+\bar{\delta}-1}{\underline{\delta}(1-\underline{\delta})}\right)\frac{F(1-\underline{\delta})}{F(1-\bar{\delta})} + \right. \\ & \left. \left(\frac{\gamma+\underline{\delta}-1}{\bar{\delta}(1-\bar{\delta})} - \frac{\gamma-\bar{\delta}}{\underline{\delta}(1-\underline{\delta}}\right)\frac{F(\underline{\delta})}{F(\bar{\delta})}\right)\end{aligned}$$

Substituting in for $F(\delta)$ and $F(1-\delta)$, collecting terms and using the facts: $\bar{\delta} > \underline{\delta}$ and $\gamma \geq 1$, we get that the term in parentheses can be rewritten as follows

$$\begin{aligned}&\gamma\left(\bar{\delta}(1-\bar{\delta})\underline{\delta}(1-\underline{\delta})\left(\frac{(1-\bar{\delta})^{2(\gamma-1)}}{\bar{\delta}^{2\gamma}} - \frac{(1-\underline{\delta})^{2(\gamma-1)}}{\underline{\delta}^{2\gamma}}\right) + \frac{(1-\underline{\delta})^{2\gamma}}{\underline{\delta}^{2(\gamma-1)}} - \frac{(1-\bar{\delta})^{2\gamma}}{\bar{\delta}^{2(\gamma-1)}}\right) + \\ &\left(\left(\bar{\delta}(1-\bar{\delta})\underline{\delta}(1-\underline{\delta})\Big(\frac{(1-\bar{\delta})^{2(\gamma-1)}}{\bar{\delta}^{2\gamma}}\underline{\delta} - \frac{(1-\underline{\delta})^{2(\gamma-1)}}{\underline{\delta}^{2\gamma}}\bar{\delta}\Big)\right) + \frac{(1-\bar{\delta})^{2\gamma+1}}{\bar{\delta}^{2(\gamma-1)}} - \frac{(1-\underline{\delta})^{2\gamma+1}}{\underline{\delta}^{2(\gamma-1)}}\right) \leq 0.\end{aligned}$$

and therefore $\widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta}^{\star};\bar{\delta}^{\star}) + \widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^{\star};\bar{\delta}^{\star}) \geq 0$ since $\gamma \geq 1$

Next, Cramer's rule implies that

$$\underline{\delta}^{\star}_{\pi^B} = \frac{\underline{\delta}^{\star}\widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^{\star};\bar{\delta}^{\star}) - \widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^{\star};\bar{\delta}^{\star})}{\det \widehat{\mathcal{V}}} > 0,$$

$$\underline{\delta}^\star_{\pi^G} = \frac{(1-\underline{\delta}^\star)\widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) + \widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)}{\det \widehat{\mathcal{V}}} > 0,$$

$$\bar{\delta}^\star_{\pi^B} = \frac{-\underline{\delta}^\star \widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)}{\det \widehat{\mathcal{V}}} > 0,$$

$$\bar{\delta}^\star_{\pi^G} = \frac{-(1-\underline{\delta}^\star)\widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)}{\det \widehat{\mathcal{V}}} > 0,$$

$$\underline{\delta}^\star_{\pi} = \frac{-\widehat{\mathcal{V}}_{\bar{\delta}\underline{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)}{\det \widehat{\mathcal{V}}} < 0$$

and

$$\bar{\delta}^\star_{\pi} = \frac{\widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)}{\det \widehat{\mathcal{V}}} < 0,$$

Observe that if

$$\underline{\delta}^\star \pi^{iB}_\mu + (1-\underline{\delta}^\star)\pi^{iG}_\mu - \pi^i_\mu \leq 0,$$

then

$$\bar{\delta}^\star_\mu = \bar{\delta}^\star_{\pi^B}\pi^{iB}_\mu(1;\mu) + \bar{\delta}^\star_{\pi^G}\pi^{iG}_\mu(1;\mu) + \bar{\delta}^\star_{\pi}\pi^{i}_\mu(0;\mu) \leq 0,$$

and $\underline{\delta}^*_\mu \lesseqgtr 0$.

If

$$\underline{\delta}^\star \pi^{iB}_\mu + (1-\underline{\delta}^\star)\pi^{iG}_\mu - \pi^i_\mu \geq 0,$$

then

$$\underline{\delta}^\star_\mu = \underline{\delta}^\star_{\pi^B}\pi^{iB}_\mu(1;\mu) + \underline{\delta}^\star_{\pi^G}\pi^{iG}_\mu(1;\mu) + \underline{\delta}^\star_{\pi}\pi^{i}_\mu(0;\mu) \leq 0.$$

and $\bar{\delta}^*_\mu \geq 0$.

Note that $\underline{\delta}^\star_\mu \leq 0$ if and only if

$$\widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)\left(\pi^{iG}_\mu - \pi^i_\mu\right) - (\underline{\delta}\widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) - \widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)(\pi^{iG}_\mu - \pi^{iB}_\mu) \geq 0$$

Because $\pi^{iG}_\mu - \pi^{iB}_\mu > \pi^{iG}_\mu - \pi^i_\mu$, then $\widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\pi^{iG}_\mu - \pi^{iB}_\mu) < \widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\pi^{iG}_\mu - \pi^i_\mu)$. Thus,

$$\begin{aligned}&\widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)\left(\pi^{iG}_\mu - \pi^i_\mu\right) - (\underline{\delta}\widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) - \widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)(\pi^{iG}_\mu - \pi^{iB}_\mu)\\ \geq&\big((1-\underline{\delta})\widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) + \widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)\big)(\pi^{iG}_\mu - \pi^{iB}_\mu)\\ \geq&\, 0\end{aligned}$$

where the last inequality follows from the fact that the second term in parethesis is positive.

Observe that

$$\bar{\delta}^\star_\mu - \underline{\delta}^\star_\mu = (\bar{\delta}^\star_{\pi^B} - \underline{\delta}^\star_{\pi^B})\pi^{iB}_\mu(1;\mu) + (\bar{\delta}^\star_{\pi^G} - \underline{\delta}^\star_{\pi^G})\pi^{iG}_\mu(1;\mu) + (\bar{\delta}^\star_{\pi} - \underline{\delta}^\star_{\pi})\pi^{i}_\mu(0;\mu) \leq 0,$$

where

$$\bar{\delta}^\star_{\pi^B} - \underline{\delta}^\star_{\pi^B} = \frac{\widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) - \underline{\delta}^\star(\widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) + \widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star))}{\det \widehat{\mathcal{V}}} < 0,$$

$$\bar{\delta}^\star_{\pi^G} - \underline{\delta}^\star_{\pi^G} = \frac{-(1-\underline{\delta}^\star)(\widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) + \widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)) - \widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)}{\det \widehat{\mathcal{V}}} > 0$$

and

$$\bar{\delta}^\star_{\pi} - \underline{\delta}^\star_{\pi} = \frac{\widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) + \widehat{\mathcal{V}}_{\bar{\delta}\underline{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)}{\det \widehat{\mathcal{V}}} < 0.$$

It readily follows from this that

$$\begin{aligned}\bar{\delta}^\star_\mu - \underline{\delta}^\star_\mu = & -\frac{\widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) + \widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)}{\det \widehat{\mathcal{V}}}\left(\underline{\delta}^\star\left(\pi^{iB}_\mu - \pi^i_\mu\right) + (1-\underline{\delta}^\star)\left(\pi^{iG}_\mu - \pi^i_\mu\right)\right) + \\ & \frac{\widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)}{\det \widehat{\mathcal{V}}}\left(\pi^{iB}_\mu - \pi^i_\mu - \left(\pi^{iG}_\mu - \pi^i_\mu\right)\right).\end{aligned}$$

Thus, $\bar{\delta}^\star_\mu - \underline{\delta}^\star_\mu \geq 0$ if and only if

$$\pi^{iG}_\mu - \pi^i_\mu \geq \frac{-\underline{\delta}^\star(\widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) + \widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)) + \widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)}{(\widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) + \widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)}(\pi^{iB}_\mu - \pi^{iG}_\mu) \tag{36}$$

The left-hand side is positive, and the rigth-hand side is negative since the ratio is positive and the term in parentheses is negative. Thus, $\bar{\delta}^\star_\mu - \underline{\delta}^\star_\mu \geq 0$.

Recall that

$$\mathbb{P}_{\bar{\delta}}(d^\star = 1) = \frac{\bar{\delta}^\star - \delta}{\bar{\delta}^\star - \underline{\delta}^\star}$$

and therefore

$$\frac{\partial \mathbb{P}_{\delta}(d^\star = 1)}{\partial \mu} = \bar{\delta}^\star_\mu \frac{\delta - \underline{\delta}^\star}{(\bar{\delta}^\star - \underline{\delta}^\star)^2} + \underline{\delta}^\star_\mu \frac{\bar{\delta}^\star - \delta}{(\bar{\delta}^\star - \underline{\delta}^\star)^2} \leq 0,$$

whenever

$$\underline{\delta}^\star\left(\pi^{iB}_\mu - \pi^i_\mu\right) + (1-\underline{\delta}^\star)\left(\pi^{iG}_\mu - \pi^i_\mu\right) \leq 0.$$

and it is positive when the opposite holds and either $\underline{\delta}^\star \geq 1/2$ or $\underline{\delta}^\star < 1/2$ and condition (**??**) holds. Otherwise, $\frac{\partial \mathbb{P}_\delta(d^\star=1)}{\partial \mu} \geq 0$ whenver

$$\frac{\pi_\mu^{iG} - \pi_\mu^i}{\left|\pi_\mu^{iB} - \pi_\mu^i\right|} \geq N(\underline{\delta}^\star, \bar{\delta}^\star)$$

$$N(\underline{\delta}^\star, \bar{\delta}^\star) \equiv -\frac{-\mathbb{P}_\delta(d^\star = 0)\underline{\delta}^\star(\widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)) + \widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) + \widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) + \underline{\delta}^\star\widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) - \widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)}{-\mathbb{P}_\delta(d^\star = 0)(1-\underline{\delta}^\star)(\widehat{\mathcal{V}}_{\underline{\delta}\underline{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) + \widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)) - \widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) + (1-\underline{\delta}^\star)\widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) + \widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star)}.$$

Recall that

$$\mathbb{E}_\delta[\tau^\star] = \mathbb{P}_\delta(d^\star = 1)\, g(\underline{\delta}^\star) + \mathbb{P}_\delta(d^\star = 0)\, g(\bar{\delta}^\star) - g(\delta), \quad \text{where} \quad g(\delta) = \frac{2\,(1-2\delta)}{\sigma^2}\ \ln\left(\frac{1-\delta}{\delta}\right).$$

It readily follows from this that

$$\frac{\partial \mathbb{E}_\delta[\tau^\star]}{\partial \mu} = \frac{\partial \mathbb{P}_\delta(d^\star = 1)}{\partial \mu}(g(\underline{\delta}^\star) - g(\bar{\delta}^\star)) + \mathbb{P}_\delta(d^\star = 1)\, g'(\underline{\delta}^\star)\underline{\delta}^\star_\mu + \mathbb{P}_\delta(d^\star = 0)\, g'(\bar{\delta}^\star)\bar{\delta}^\star_\mu,$$

where

$$g'(\delta) = \frac{2}{\sigma^2}\left(-2\ln\frac{1-\delta}{\delta} - \frac{1-2\delta}{\delta(1-\delta)}\right).$$

Thus, $g'(\delta) \leq 0$ for $\delta \leq 1/2$ and $g'(\delta) > 0$ otherwise and $g''(\delta) \geq 0$. In addition, $g(\underline{\delta}) - g(\bar{\delta}) \geq 0$ if $\bar{\delta} \leq 1/2$, $g(\underline{\delta}) - g(\bar{\delta}) < 0$ if $\underline{\delta} \geq 1/2$, and $g(\underline{\delta}) - g(\bar{\delta}) \lesseqgtr 0$ if $\underline{\delta} \leq 1/2 < \bar{\delta}$.

Observe that

$$\frac{\partial \mathbb{E}_\delta[\tau^\star]}{\partial \mu} = \mathbb{P}_\delta(d^\star = 0)\bar{\delta}^\star_\mu\left(g'(\bar{\delta}^\star) - \frac{g(\bar{\delta}^\star) - g(\underline{\delta}^\star)}{\bar{\delta}^\star - \underline{\delta}^\star}\right) + \mathbb{P}_\delta(d^\star = 1)\underline{\delta}^\star_\mu\left(g'(\underline{\delta}^\star) - \frac{g(\bar{\delta}^\star) - g(\underline{\delta}^\star)}{\bar{\delta}^\star - \underline{\delta}^\star}\right).$$

This can be written as follows

$$\begin{aligned}\frac{\partial \mathbb{E}_\delta[\tau^\star]}{\partial \mu} =& \bar{\delta}^\star_\mu\left(g'(\bar{\delta}^\star) - \frac{g(\bar{\delta}^\star) - g(\underline{\delta}^\star)}{\bar{\delta}^\star - \underline{\delta}^\star}\right) + \\ & \mathbb{P}_\delta(d^\star = 1)\left(\underline{\delta}^\star_\mu\left(g'(\underline{\delta}^\star) - \frac{g(\bar{\delta}^\star) - g(\underline{\delta}^\star)}{\bar{\delta}^\star - \underline{\delta}^\star}\right) - \bar{\delta}^\star_\mu\left(g'(\bar{\delta}^\star) - \frac{g(\bar{\delta}^\star) - g(\underline{\delta}^\star)}{\bar{\delta}^\star - \underline{\delta}^\star}\right)\right).\end{aligned}$$

It follows from this that if $\underline{\delta}_\mu \leq 0$ and $\underline{\delta}_\mu \leq 0$

$$\frac{\partial \mathbb{E}_\delta[\tau^\star]}{\partial \mu} \geq 0 \implies \mathbb{P}_\delta(d^\star = 1) \geq \frac{1}{1-A}$$

where

$$A \equiv \frac{\underline{\delta}^\star_\mu \left(g'(\underline{\delta}^\star) - \frac{g(\bar{\delta}^\star) - g(\underline{\delta}^\star)}{\bar{\delta}^\star - \underline{\delta}^\star} \right)}{\bar{\delta}^\star_\mu \left(g'(\bar{\delta}^\star) - \frac{g(\bar{\delta}^\star) - g(\underline{\delta}^\star)}{\bar{\delta}^\star - \underline{\delta}^\star} \right)} \leq 0.$$

Substituting for the value $\mathbb{P}_\delta(d^\star = 1)$ and the grouping terms, we get that this holds whenever $A(\underline{\delta}^\star - \bar{\delta}^\star) \geq 0$. Thus, $\mathbb{E}_\delta[\tau^\star]$ rises with μ.

Because $g(\cdot)$ is strictly convex and $\bar{\delta}^\star > \underline{\delta}^\star$, the first term in parenthesis is positive and the second is negative, if $\underline{\delta}^\star_\mu \geq 0$, and $\bar{\delta}^\star_\mu \leq 0$, the expected time falls with μ. This happens whenever

$$\frac{\underline{\delta}^\star \pi^{iB}_\mu + (1 - \underline{\delta}^\star)\pi^{iG}_\mu}{\pi^i_\mu} < \frac{\underline{\delta}^\star \pi^{iB} + (1 - \underline{\delta}^\star)\pi^{iG}}{\pi^i}.$$

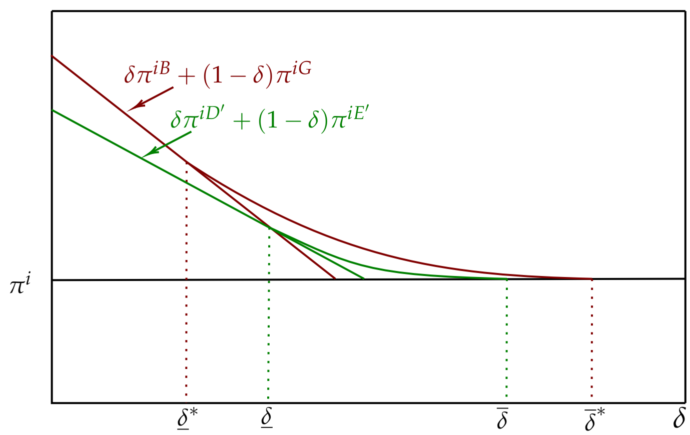

Figure 2: Comparative Statics

Proof of Proposition 7: Value matching and smooth-pasting at $\underline{\delta}^\star$ entail the following

$$\widehat{\mathcal{V}}(\underline{\delta}^\star; \bar{\delta}^\star)\mathcal{V}(\eta;\rho) = \delta^\star \pi^{iB}(1;\mu) + (1-\delta^\star)\pi^{iG}(1;\mu)$$

$$\widehat{\mathcal{V}}_\delta(\underline{\delta}^\star; \bar{\delta}^\star)\mathcal{V}(\eta;\rho) = \pi^{iB}(1;\mu) - \pi^{iG}(1;\mu)$$

Totally differentiating both equations with respect to $\bar{\delta}$, $\underline{\delta}$, and γ, we get that

$$\begin{pmatrix} \mathcal{V}(\eta;\rho)\widehat{\mathcal{V}}_{\underline{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) - \triangle_\pi(\mu) & \mathcal{V}(\eta;\rho)\widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) \\ \mathcal{V}(\eta;\rho)\widehat{\mathcal{V}}_{\underline{\delta}\delta}(\underline{\delta}^\star;\bar{\delta}^\star) & \mathcal{V}(\eta;\rho)\widehat{\mathcal{V}}_{\delta\bar{\delta}}(\underline{\delta}^\star;\bar{\delta}^\star) \end{pmatrix} \times \begin{pmatrix} \underline{\delta}^\star_{\gamma(\rho)} \\ \underline{\delta}^\star_{\gamma(\rho)} \end{pmatrix} = \begin{pmatrix} -\mathcal{V}_{\gamma(\rho)(\eta;\rho)}\widehat{\mathcal{V}}(\underline{\delta}^\star;\bar{\delta}^\star) \\ -\mathcal{V}_{\gamma(\rho)(\eta;\rho)}\widehat{\mathcal{V}}_\delta(\underline{\delta}^\star;\bar{\delta}^\star) \end{pmatrix}$$

where

$$\mathcal{V}_{\gamma(\rho)}(\eta;\rho) \geq 0$$

and the sign follows from the following facts: i) $\frac{(\gamma-\bar{\eta})}{(2\gamma-1)}$ rises and $\frac{(\gamma+\bar{\eta}-1)}{(2\gamma-1)}$ falls with γ whenever $\bar{\eta} \geq 1/2$; and ii)

$$\partial\left(\frac{F(\eta)}{F(\bar{\eta})}\right) \Big/ \partial\gamma = \frac{F(\eta)}{F(\bar{\eta})} \ln\left(\frac{1-\eta}{\eta}\frac{\bar{\eta}}{1-\bar{\eta}}\right) > 0$$

and

$$\partial\left(\frac{F(1-\eta)}{F(1-\bar{\eta})}\right) \Big/ \partial\gamma = -\frac{F(1-\eta)}{F(1-\bar{\eta})} \ln\left(\frac{1-\eta}{\eta}\frac{\bar{\eta}}{1-\bar{\eta}}\right) < 0.$$

Next, Cramer's rule implies that

$$\underline{\delta}^{\star}_{\gamma(\rho)} = \frac{-\widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^{\star};\bar{\delta}^{\star}) + \widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^{\star};\bar{\delta}^{\star})}{\det \widehat{\mathcal{V}}} \mathcal{V}_{\gamma(\rho)}(\eta;\rho)\mathcal{V}(\eta;\rho)\widehat{\mathcal{V}}(\underline{\delta}^{\star};\bar{\delta}^{\star}) \leq 0$$

and

$$\bar{\delta}^{\star}_{\gamma(\rho)} = \frac{\widehat{\mathcal{V}}_{\underline{\delta\delta}}(\underline{\delta}^{\star};\bar{\delta}^{\star})}{\det \widehat{\mathcal{V}}} \mathcal{V}_{\gamma(\rho)}(\eta;\rho)\mathcal{V}(\eta;\rho)Sa(\underline{\delta}^{\star};\bar{\delta}^{\star}) \leq 0$$

Thus,

$$\bar{\delta}^{\star}_{\gamma(\rho)} - \underline{\delta}^{\star}_{\gamma(\rho)} = \frac{\widehat{\mathcal{V}}_{\underline{\delta}\bar{\delta}}(\underline{\delta}^{\star};\bar{\delta}^{\star}) + \widehat{\mathcal{V}}_{\underline{\delta\delta}}(\underline{\delta}^{\star};\bar{\delta}^{\star}) - \widehat{\mathcal{V}}_{\bar{\delta}}(\underline{\delta}^{\star};\bar{\delta}^{\star})}{\det \widehat{\mathcal{V}}} \mathcal{V}_{\gamma(\rho)}(\eta;\rho)\mathcal{V}(\eta;\rho)\widehat{\mathcal{V}}(\underline{\delta}^{\star};\bar{\delta}^{\star}) \leq 0$$

$\square$

PROOF OF PROPOSITION 10. Observe that

$$\begin{aligned}
&\mathbb{E}_{\tau_2,d_2,\delta_2}\left[e^{-r\,\tau_2(0,G)}\left(\pi^{1\theta_2}(0,d_2(0,G);\mu) - \pi^{1}(0,0;\mu)\right)\right] + c \\
=&\mathbb{E}_{\tau_2}\left[e^{-r\,\tau_2(0,G)}\right]P(d_2(0,G)=1)\big(\delta_2\pi^{1B}(0,1;\mu) + (1-\delta_2)\pi^{1G}(0,1;\mu) - \pi^{1}(0,0;\mu)\big) + c \\
\geq&0 \\
\iff& \delta_2 \geq \frac{\pi^{1}(0,0;\mu) - \pi^{1G}(0,1;\mu) - c}{\pi^{1B}(0,1;\mu) - \pi^{1G}(0,1;\mu)}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_{\tau_2,d_2,\delta_2}\left[e^{-r\,\tau_2(1,B)}\left(\pi^{1B\theta_2}(1,d_2(1,B);\mu) - \pi^{1B}(1,0;\mu)\right)\right] + \pi^{1B}(1,0;\mu)) - \\
&\mathbb{E}_{\tau_2,d_2,\delta_2}\left[e^{-r\,\tau_2(1,G)}\left(\pi^{1G\theta_2}(1,d_2(1,G);\mu) - \pi^{1G}(1,0;\mu)\right)\right] - \pi^{1G}(1,0;\mu)) \leq 0 \\
&\iff \\
&\mathbb{E}_{\delta_2}\left[e^{-r\,\tau_2(1,B)}\right]P(d_2(1,B)=1)\big(\delta_2\pi^{1BB}(1,1;\mu) + (1-\delta_2)\pi^{1BG}(1,1;\mu) - \pi^{1B}(1,0;\mu)\big) + \pi^{1B}(1,0;\mu)) - \\
&\mathbb{E}_{\delta_2}\left[e^{-r\,\tau_2(1,G)}\right]P(d_2(1,G)=1)\big(\delta_2\pi^{1GB}(1,1;\mu) + (1-\delta_2)\pi^{1GG}(1,1;\mu) - \pi^{1G}(1,0;\mu)\big) - \pi^{1G}(1,0;\mu)) \leq 0,
\end{aligned}$$

where

$$\mathbb{P}_\delta(d_2(d_1,\theta_1)=1)=\frac{\bar{\delta}_2(d_1,\theta_1)-\delta_2}{\bar{\delta}_2(d_1,\theta_1)-\underline{\delta}_2(d_1,\theta_1)}>0$$

for all $\delta_2 \leq \bar{\delta}_2(d_1,\theta_1)$.

$$\begin{aligned}
&\mathbb{E}_{\tau_2,d_2,\delta_2,\delta_1}\left[e^{-r\,\tau_2(1,\theta_1)}\left(\pi^{1\theta}(1,d_2(1,\theta_1);\mu)-\pi^{1\theta_1}(1,0;\mu)\right)\right]+\\
&\delta_1\pi^{1B}(1,0;\mu)+(1-\delta_1)\pi^{1G}(1,0;\mu)-\pi^1(0,0;\mu)+c=\\
&\delta_1\mathbb{E}_{\delta_2}\left[e^{-r\,\tau_2(1,B)}\right]P(d_2(1,B)=1)\left(\delta_2\pi^{1BB}(1,1;\mu)+(1-\delta_2)\pi^{1BG}(1,1;\mu)-\pi^{1B}(1,0;\mu)\right)+\\
&(1-\delta_1)\mathbb{E}_{\delta_2}\left[e^{-r\,\tau_2(1,G)}\right]P(d_2(1,G)=1)\left(\delta_2\pi^{1GB}(1,1;\mu)+(1-\delta_2)\pi^{1GG}(1,1;\mu)-\pi^{1G}(1,0;\mu)\right)\Big)+\\
&\delta_1\pi^{1B}(1,0;\mu)+(1-\delta_1)\pi^{1G}(1,0;\mu)-\pi^1(0,0;\mu)+c\geq 0
\end{aligned}$$

Substituting these back into the smooth-pasting and value-matching conditions, comparing them with the ones in equations (12) and (13), and simplifying, we deduce the result.

$\square$