# Competition Between Private and Public Providers with Vouchers

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**Abstract**

Over the last decade, many governments have allowed private providers, partially funded through vouchers, to enter markets previously supplied solely by public providers, such as education, security, and health. Many argued that introducing subsidized competition would improve matching between customers and providers, as well as the quality provided by both private and public providers at affordable prices. This article argues that, in the presence of both vertical and horizontal differentiation, the relationship between prices and quality is much more nuanced. This, together with firm heterogeneity, imposes stringent constraints on demand, making it difficult to confidently justify a monotonic relationship among prices, quality, and vouchers.

**Key Words:** Vouchers, Public Providers, Private Providers, Quality, Market Power, vertical differentiation, horizontal differentiation.

**JEL Codes:** D2, D4, H52, I22, I20, L1

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# 1 Introduction

Governments face growing pressure to deliver high-quality public services at reasonable prices. In response, many countries have opened sectors historically provided by the state to private competition and introduced vouchers or per-user subsidies intended to give consumers choice and to harness market incentives for quality and efficiency. This shift—apparent across education, health, transport, waste management, and security—rests on a simple normative intuition: exposing public providers to private rivals and funding user choices with vouchers will foster responsiveness, spur quality improvements, and contain costs. This view has strong adherents. Yet, it also provokes intense controversy: critics argue private providers may prioritize profits at the expense of quality, cream-skim the most desirable users, and exacerbate inequality. Empirically, there are positive and negative experiences.[1]

This article addresses a fundamental question: how does the presence of private providers, partially financed with vouchers, in markets previously supplied exclusively by public providers affect prices and quality? An intuitive and plausible answer is that introducing private providers should give customers more options and thereby intensify competition. As a result, providers should supply better-quality goods and services. In addition, if private providers are partially financed with vouchers, the higher quality should have little impact on prices because of the negative pass-through from vouchers to prices.

We answer this question by combining the classical model of horizontal differentiation of Perloff and Salop (1985) with the vertical differentiation model of Shaked and Sutton (1982). When firms set the same price, under horizontal differentiation, each firm faces positive demand, whereas under vertical differentiation, all consumers prefer the higher-quality product. Consumers are heterogeneous in their marginal valuation of quality and have idiosyncratic non-pecuniary preferences over providers; firms compete on quality first and on price second. Public providers' objectives are a convex combination of market share and profits, whereas private providers maximize profits only. This setup captures institutional asymmetries while retaining standard microfoundations for choice and

[1]The empirical literature is discussed in Section 6, following the model results.

firm incentives and reflects markets where the competition between private and public providers compete, as in health, education, and security.

When the horizontal differentiation dimension is not considered, the intuitive answer given earlier remains valid. In the classical vertical model of Shaked and Sutton (1982) with one public and one private provider, providers must differentiate through quality choices to avoid a Bertrand-like competition, in which both set prices equal to marginal costs. This implies that if one provider chooses a higher quality, the competitor sets a lower price; otherwise, every customer will patronize the private provider. When quality valuations are uniformly distributed, and we focus on the full coverage equilibrium, prices decrease with the voucher and the per-customer subsidy, and rise with private and public quality.[2] This happens because as a provider raises the price, its competitor can be less aggressive, i.e., raise the price, and not lose too many customers, and a provider wishes to set a higher price because higher quality implies a larger marginal cost. Holding quality constant, the pass-through from the voucher and the subsidy to prices is lower than 1. Thus, the private provider's markup rises and the public provider's markup falls with the voucher, and the opposite happens with the subsidy. The pass-through from a higher marginal cost due to a higher quality is also lower than 1. Because an increase in a provider's quality increases demand less, the higher the competitors' quality, the more the private provider's quality increases with the voucher, and the more the public provider's quality decreases. Prices may rise or fall depending on the size of the pass-through from quality to prices relative to that from vouchers to prices. Public prices are lower because they are concerned with market share, and quality falls less than it should because their objective function places a positive weight on market share. However, a larger voucher makes prices less sensitive to quality. Thus, in this setting, introducing competition with vouchers favors those patronizing the private provider and harms those in the public provider. This effect can be mitigated by increasing the per-customer subsidy.[3]

The paper makes three broad empirical-theoretical points. First, when consumers

[2]Wauthy (1996) shows that when consumers are concentrated, there is fierce competition for them that even the choice of quality cannot avoid. This will result in high-quality goods in both firms and little dispersion between them. When consumers are more dispersed, the result is more nuanced.

[3]A higher voucher results in more customers patronizing the private provider. Thus, the harm caused by lower public quality decreases as the voucher rises.

are heterogeneous and markets exhibit both vertical and horizontal differentiation, the monotonically increasing prices with quality that arise in symmetric or purely vertical models break down. Because customers rank providers with equal quality and prices differently, it is hard to characterize providers' equilibrium behavior. This makes it difficult to pin down the sign and magnitude of the strategic-commitment effect of quality, which is the main determinant of it. Common modeling shortcuts—symmetric firms, homogeneous quality valuations, or single-dimension differentiation—deliver tractable, often monotone predictions (e.g., quality always increases prices). Those predictions are fragile: they follow from structure imposed by symmetry and single-dimensional differentiation rather than from robust economic forces. Second, allowing consumer heterogeneity in quality valuation and a richer two-dimensional differentiation structure yields a complex stratification pattern, making the analysis difficult. For instance, in the standard Salop model, adding a quality choice stage before prices without consumer heterogeneity yields the standard result: the higher the quality, the higher the price relationship.[4].Third, allowing competition within each sector is important because a voucher only affects directly private providers' markups. Similarly, for the subsidy. When there is only one provider of each type, an increase in the voucher (subsidy) raises the private (public) provider's markup. In contrast, when there is competition within and between sectors, it may not. The impact of the voucher on the markup, which is crucial to understanding the quality choice, is hard to determine.

The equilibrium qualities are determined by the business-stealing effect, which captures how many new customers a firm attracts by increasing quality when competitors' quality and prices are held constant; the strategic-commitment effect, which measures the demand variation due to the competitors' optimal price responses to an increase in quality. When quality decreases competitors' prices, demand increases and vice-versa; and the cost effect, which measure the impact of quality on total costs. The first two effects are partially determined by the fact that new customers are not a random sample of customers patronizing competitors, but rather are those whose willingness to pay for the quality increase supplied by a given firm is the highest.

[4]See, Vogel (2008)

The change in the sub-game perfect equilibrium prices and qualities depends on the impact of vouchers on these effects, which is, in general, ambiguous, and in many cases, of opposing direction. This raises doubts about the ability of the policy of introducing private competition with vouchers to serve as a proper mechanism for supplying high-quality goods and services at reasonable prices.[5]

More concretely, the model yields two central insights. One, holding quality levels constant, prices decrease monotonically with the voucher and public subsidy, but need not do so with quality.

Prices fall with the voucher because a higher voucher, holding prices constant, increases the markup, and therefore private providers wish to attract more customers. To do so, they lower prices. Public providers follow through, and this induces everyone to lower prices further (prices are strategic complements). Similarly, for an increase in the subsidy.

An increase in quality raises the marginal cost of serving a customer and decreases the demand elasticity since willingness to pay increases. Because a higher quality lowers the markup, the firm wishes to sell less and raise prices. Because prices are strategic complements, everyone responds by being less aggressive, i.e., increasing prices, which ameliorates the decrease in demand. At the same time, the increase in quality, holding prices constant, reduces competitors' demand. This induces them to lower prices, reinforcing a broader price decline. Thus, the first effect pushes prices up, while the second pushes them down. When the direct effects dominate the partial effects, i.e., the standard dominant diagonal condition holds, the firm supplying higher quality sets a higher price; competitors may either set a higher or lower price. Thus, in general, a higher quality may result in higher or smaller prices.

The discussion above has two consequences. First, the business-stealing effect might be small because the price differences across firms reduce the slope of the demand with respect to quality. Second, the strategic-commitment effect of quality could be positive or negative. Because some competitors may respond by increasing their prices and others

[5]This could be a good policy to increase coverage. We focus on a full-coverage case to isolate the impact of vouchers on prices and quality from the issue of increased participation. When possible, we comment on the partial coverage case.

by decreasing them when faced with a quality increase by any provider, the demand of the corresponding provider may either rise or fall with quality. Because higher vouchers (subsidies) induce private providers to act more aggressively than public providers, it is more likely that the business-stealing of quality is small, and the strategic commitment effect of quality could be negative, or they have the opposite sign. In equilibrium, if quality is positive, the sum of the business-stealing and strategic-commitment effect must be positive.

When a weaker version of the standard dominant diagonal condition holds, we have the following results:

- Prices set by the regulatory authority. When qualities are strategic complements, which occurs when the demand faced by each firm is concave in its quality, quality rises with the voucher. This never happens when competition, measured by the number of firms, is intense. When qualities are strategic substitutes, i.e., demand is convex in quality, there is at least one private provider that offers a higher quality when the voucher increases and a public provider whose quality decreases. If firms are symmetric, all private providers offer higher quality, and all public providers offer lower quality. This increases stratification across sectors. In both cases, aggregated quality increases with the voucher (subsidy).

- Prices are strategically chosen. If the profit gain from increasing quality rises with the voucher, then at least one private provider increases its quality as the voucher rises. Suppose qualities are strategic substitutes, which is more likely to be the case since the direct effect tends to dominate equilibrium effects. In that case, this implies that at least one public provider lowers its quality. This increases stratification. Aggregated quality increases with the voucher (subsidy) whenever the profit gain from increasing quality is non-decreasing with the voucher (subsidy) for both private and public providers. The results are similar when public providers' prices are set, whereas private providers are free to set their own prices.

Taken together, the results demonstrate that the claim that competition and vouchers either increase quality is the result of simplifying modeling choices (symmetry, onedimensional differentiation, homogeneous valuations) rather than more realistic differentiation and firm heterogeneity, due at least to different objective functions by private and public providers. The remainder of the paper constructs the formal model, characterizes the equilibrium, and derives comparative statics.

Policy implications follow naturally. Policymakers who rely on symmetric or vertically-focused models risk overconfident prescriptions. The often-repeated motto that choice is "the tide that raises all boats" requires strong and specific conditions on demand curvature, strategic complementarities, and the distribution of consumers' idiosyncratic preferences; absent those, introducing private competition funded by vouchers may produce ambiguous or adverse outcomes for prices, quality, and selection. Thus, careful market diagnosis — assessing heterogeneity in valuations, the strength of horizontal preference dispersion, monitoring capacity, and the likely responses of incumbent public providers — is a necessary precondition for non-discriminatory voucher-based reforms that aim to raise quality rather than merely expand coverage.

The rest of the paper is as follows. In the next section, we discuss the related literature. In Section 3, we present the model. After that, in Section 4, we derive the pricing sub-game and the main comparative statics. In the next Section, we study the sub-game perfect equilibrium and study the comparative statics considering equilibrium qualities and vouchers (public provider subsidies). In Section 6, we discuss the evidence in the schooling and health markets. In the last Section, we present some concluding remarks.

## 2 Related Literature

This paper contributes to and draws upon three broad literatures: product differentiation and quality-price competition, mixed oligopoly models of public–private competition, and the voucher/school-choice literature in education (and related work on vouchers in health). Below, we position our contribution relative to key strands of work and highlight complementarities and departures.

The canonical quality-price frameworks of Gabszewicz and Thisse (1979, 1986) and Shaked and Sutton (1982, 1983) analyze strategic quality choice followed by price competition in vertically differentiated markets. Those contributions clarify how quality investments and the resulting cost structure interact with price-setting—insights we retain. Vogel (2008) investigates horizontal-vertical in the Salop circle with homogeneous quality valuations, finding that quality behaves as a function of marginal costs rather than as a strategic lever—an outcome driven by additive separability and valuation homogeneity.

We augment these approaches by embedding horizontal differentiation à-la-Perloff and Salop (1985) and vertical differentiation à-la-Shaked and Sutton (1982), which breaks the separability assumptions often used to simplify comparative statics. In this setting, quality is a key strategic instrument that firms can use to mitigate price competition and sort consumers across public and private providers, resulting in stratification.

A separate strand, called mixed oligopolies, analyzes markets with both public and private firms, often focusing on how public ownership affects welfare, pricing, quality, or entry incentives Cremer, Marchand, and Thisse (1989, 1991), Grilo (1994), Matsumura and Matsushima (2004). De Fraja and Delbono (1990) use a duopoly model to show the public firm chooses the lower quality while the private one choose the higher quality. Matsumura and Matsushima (2004) show that private firms make investment that allows them to operate at lower costs than the public firm. Laine and Ma (2017), also study a vertical differentiation and quality-then-price game with a single public and single private firm and show multiple equilibria in quality, where one of the two firms offer higher quality, driven by strategic substitutability in qualities.

Most models assume away competition within each sector, a single dimension of differentiation, or no differentiation (typically Cournot), and a free public firm that maximizes social welfare (or some transformation thereof) while the private firm maximizes profit. Our model departs by allowing competition by multiple, possibly heterogeneous firms within each sector, by combining vertical and horizontal differentiation, and by allowing the public providers to choose prices and qualities strategically. These extensions are needed to better capture the main characteristics of the markets where private and public providers compete. In addition, the monotonicity conditions found in symmetric mixed-oligopoly models need not hold when heterogeneity and horizontal differentiation are present.

The voucher and school-choice literature in economics and public policy is vast.[6] Friedman (1955, 1962) argued vouchers are a mechanism to improve choice and matching between family preferences and schools. Formal and empirical work has since underscored complex distributional and strategic consequences. Epple and Romano (1998, 2008) characterize sorting and cream-skimming in models with heterogeneous households and peer preferences. Nechyba (1999, 2000, 2003) and Ferreyra (2007) extend these insights in spatial general-equilibrium simulations. A common thread in much of this literature is that quality differences often reflect student selection and peer effects rather than strategic quality investments: when quality is determined primarily by the composition of enrolled students -peer effects-, competition has different implications than when schools strategically invest in quality. Our paper brings attention to that distinction by modeling quality as a strategic investment that raises marginal costs and interacts with consumer heterogeneity and horizontal preference dispersion. McMillan (2004) analyzes public quality responses to private entry under free entry and shows that vouchers can lower public quality through intensified competition for high-valuation customers.

The technical literature on monotone comparative statics and games with strategic complementarities provides tools for deriving unambiguous comparative statics predictions (Vives, 2009). While our setting is such that prices are strategic complements under log-concave distributions, in general, strategic complementarity in qualities cannot be guaranteed. Therefore, we have to draw on sufficient conditions in the spirit of the standard dominant diagonal condition. Namely, $B_0$-matrices, introduced by Christensen (2018), which are demanding with multidimensional differentiation, but less so than the standard dominant diagonal condition. We build on these insights: when the $B_0$ matrix-type conditions and convexity of taste distributions hold, monotone comparative statics are possible. However, we emphasize that such a structure is restrictive, and in its absence, comparative statics are generally non-monotone.[7]

Our modeling choices are motivated by empirical patterns in education and health markets where vouchers and private competition are widespread and where outcomes

[6] The empirical literature is discussed in Section 6.

[7] This paper is also related to the literature of pass-through in oligopolies (Weyl and Fabinger, 2013, Ritz, 2024). They also impose symmetry and implicitly the dominant diagonal condition.

are heterogeneous across contexts (see Section 6 for evidence on education and health markets). The ambiguous empirical findings—sometimes yielding positive effects on access and responsiveness, sometimes leading to increased segregation or mixed quality outcomes—are consistent with our theoretical claim that heterogeneity and horizontal and vertical differentiation complicate straightforward policy recommendations.

In sum, this paper integrates quality-then-price strategic interaction, horizontal and vertical differentiation, and asymmetric public/private objectives into a unified framework. Doing so reveals that the common practice of relying on symmetric, vertically-focused models to evaluate quality and private competition can be misleading: those models impose a structure that makes comparative statics monotone, whereas realistic and necessary enhancements break that monotonicity and yield richer, context-dependent predictions.

# 3 The Environment

We consider a market for goods or services with three types of agents: customers, each consuming one unit of the good; the private sector; and the public sector (denoted by superscript 0).

**Providers** There are $n$ private providers indexed by $j \in \mathcal{J} \equiv \{1, \ldots, n\}$ and $N$ public providers indexed by $j \in \mathcal{J}^0 \equiv \{n+1, \ldots, n+N\}$. When a customer patronizes a private provider, it receives a voucher worth $v \in \Re_+$. When it patronizes a public provider, it receives a per-customer subsidy of $g \in \Re_+$.

Provider $j$'s total cost of serving $s^j$ customers when its quality is $q^j$ is $\mathcal{C}^j(s^j, q^j) = c^j(q^j)s^j + C^j(q^j)$.[8] This means that for a given quality level $q^j$, the marginal cost of serving a customer is constant and equals to $c^j(q^j)$, where $c^j(\cdot)$ is non-negative, strictly increasing and convex, with $c^j(\underline{q}) \geq 0$, $c^j_{q^j}(\underline{q}) = 0$ and $\lim_{q^j \to \infty} c^j(q^j) \to \infty$. For a given quality level, there is also a fixed cost of production which is $C^j(q^j)$, where $C^j(\cdot)$ is non-negative, strictly increasing and convex in $q^j$, with $C^j(\underline{q}) \geq 0$, $C^j_{q^j}(\underline{q}) = 0$ and $\lim_{q^j \to \infty} C^j(q^j) \to \infty$. Thus,

[8]This implies that this is a endogenous sunk-cost model.

for a given quality level, the average cost decreases as the number of customers increases, and the production technology exhibits economies of scale. We can think of quality as requiring investments in fixed inputs, such as capital goods, and variable inputs, like labor or more skilled labor.

Because we remain agnostic about whether producing goods and services is more or less expensive in the private sector than in the public sector, we have assumed that the total cost of serving any given number of customers at any given quality level by a public provider is the same as that by a private provider. When the cost of labor and capital determines the cost of quality, and agency problems are equally severe in the private and public sectors, this is the proper assumption.

Private providers aim to maximize profits, while public providers aim to maximize a weighted mean of profits and the demand (market share) they capture, as in Barseghyan, Clark, and Coate (2019). They assign a weight $\beta$ to profits and $1-\beta$ to demand. Thus, public providers are partially rent-seekers. This stacks the deck against finding that incentives have perverse effects, as we consider the case in which incentives might be expected to be needed. Suppose they chose instead to maximize quality or only market share. In that case, it is unlikely that incentives would have any efficiency effects when there is competition between private and public providers. Public providers can earn rents by charging positive prices when needed. Because public providers stand to lose funding when the number of served customers falls, on the margin they have an incentive to retain customers in the face of competition from private providers. It is also common to see public providers rewarded, at least in part, not by their performance but by their enrollment or by the share of the corresponding population they serve.[9]

**Customers** Customers have unit demands, and the value of their outside option is normalized to zero. We model horizontal product differentiation by adopting a random-

[9]Public providers could hold many other plausible objective functions, such as a weighted average between profits and customers' welfare, industry-wide quality, understood as the sum of each private and public provider's quality, weighed by its corresponding demand, or public quality, weighed by the public sector demand. We have chosen to maximize the weighted average of profits and demand because we wish to provide the strongest possible case for the argument that choice and vouchers increase competition among private and public providers, and that the latter respond by improving quality.

utility framework in the spirit of Perloff and Salop (1985). The utility of a customer when patronizing a provider that sets a price $p$ and offers quality $q$ is given by: $U(y,q,p,\theta)+\epsilon$, where $U(y,q,p,\theta) \equiv y+\theta q-p$, where $y$ is the utility of its outside option, $\theta$ is the marginal valuation for quality, and $\epsilon$ is a non-pecuniary random utility shock that is specific to each provider. We assume that $\epsilon^j$ is independent and identically distributed (i.i.d.) across individuals, which reflects idiosyncratic tastes for different firms. For a given customer, it is also independent and identically distributed (i.i.d.) across firms. $\epsilon^j$ is distributed $G(\cdot)$ with compact and full support $[\underline{\epsilon},\bar{\epsilon}] \subset \Re$, and zero mean. $g(\cdot)$ is twice continuously differentiable everywhere, bounded, log-concave, and $g'(\cdot)$ is bounded. Quality valuation $\theta$ is distributed with cumulative distribution function $F(\theta)$ with full support $\Theta \equiv [\theta_L,\theta_H]$, density $f(\theta)$, and mean $\bar{\theta}$. $f(\theta)$ is twice continuously differentiable everywhere, and log-concave.

The functional form of $U(y,q,p,\theta)$ assumes the following: first, as in Mussa and Rosen (1978), utility is linear in quality, and quality and customers' valuations are complements. Thus, each customer has a different marginal valuation of quality; second, as in Economides (1986), the utility function is additive in the two dimensions of differentiation. These two things imply that increases in quality are valued equally by customers, regardless of location. This is a standard assumption in the differentiation literature that allows the two dimensions of differentiation to be identified (Anderson, de Palma, and Thise (1992), Economides (1993), and Vogel (2008)); third, conditional on the valuation level, utility exhibits constant marginal utility of quality; and fourth, the marginal utility of quality and income are independent.

**Timing** At stage 1, both public and private providers simultaneously choose their quality levels. At Stage 2, after firms and customers observe the quality profile, public and private providers simultaneously determine prices. At the final stage, customers observe the realization of their non-pecuniary utility shock from patronizing each firm, the quality and price profile, and choose a provider. All households buy goods or services from one of the available providers.

# 4 The Pricing Sub-Game

## 4.1 Demand Characterization

Once a customer learns his random-utility shocks, he chooses the provider with the highest utility. Thus, the customer chooses provider $j \in \mathcal{J}$ whenever $U(y, q^j, p^j, \theta) + \epsilon^j \geq U(y, q^k, p^k, \theta) + \epsilon^k$ for all $k \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}$. Hence, the demand for provider $j$ is given by

$$D^j(p, q) = \mathbb{P}[U(y, q^j, p^j, \theta) + \epsilon^j \geq \max\{0, \max_{k \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}} \{U(y, q^k, p^k, \theta) + \epsilon^k\}]$$

It readily follows from this that[10]

$$D^j(p, q) = \mathbb{E}_{\theta, \epsilon}\Big[\prod_{k \in J \cup \mathcal{J}^0 \setminus \{j\}} G^k\Big(\triangle U_{jk}(\theta) + \epsilon^j\Big)\Big]$$

where the equality follows from the independence assumption about the $G$s distributions and where $q \equiv (q^1, \ldots, q^n, q^{n+1}, \ldots, q^{n+N})$ and $p \equiv (p^1, \ldots, p^n, p^{n+1}, \ldots, p^{n+N})$.

**Proposition 1.** *For any $(p, q) \in \Re_+^{2(n+N)}$,*

*i) $D^j(p, q)$ is decreasing and log-concave in $p^j$, increasing in $p^{-j}$, and log-supermodular in $p$.*

*ii) $D^j(p, q)$ is increasing and log-concave in $q^j$, decreasing in $q^{-j}$, log-submodular in $q$, and log-supermodular in $(p^j, q^j)$ and log-sub-modular in $(p^j, q^k)$ for $j \neq k$.*

The log-concavity of the provider-specific demand implies that the price elasticity of demand decreases with its price. The log-supermodularity in $p$ means that the price elasticity of demand decreases as competitors' prices increase. The latter will imply an increasing best-response correspondence when marginal costs are constant or convex and goods are gross substitutes.

[10] We will assume that $y$ is such that the utility is always positive. Thus, we focus on an equilibrium in which $(p, q)$ are such that the whole market is covered. This is a common assumption in the literature (Gabaix, Laibson, Li, Li, Resnick, and de Vries, 2016, Perloff and Salop, 1985, Anderson, Erkal, and Piccinin, 2020). Quint (2014) shows that the equilibrium is also unique when the market is not fully covered. We do not consider this case, because we wish to emphasize the cream-skimming incentives generated by quality, which requires holding the participation incentives constant.

## 4.2 Equilibrium Prices

Let $\Pi^j(p,q;\beta^j) \equiv \big(\beta^j(p^j+s-c^j(q^j))+1-\beta^j\big)D^j(p,q)$, where $(s,\beta^j)=(v,0)$ for all $j\in\mathcal{J}$; i.e., when provider $j$ is private, and $(s,\beta^j)=(g,\beta)$ for all $j\in\mathcal{J}^0$; i.e., when provider $j$ is public. Provider $j$'s goal is to maximize $\Pi^j(p,q;\beta^j)-C^j(q^j)$ with respect to $p^j$, but since $C^j(q^j)$ is independent of $p^j$, it faces the following monotonically transformed optimization problem

$$\max_{p^j\in\Re_+}\{\log\Pi^j(p,q;\beta^j)\}.$$

Provider $j$'s first-order condition, when the price is positive, is given by

$$\frac{p^j(q)+g-c^j(q^j)}{p^j(q)} = -\frac{1}{\eta^j(p(q),q)} - \frac{1-\beta^j}{\beta^j}\frac{1}{p^j(q)}, \tag{1}$$

where $\eta_j(\cdot)$ is the price elasticity of demand given by

$$\eta^j(p,q) = -\frac{p^j D^j_j(p,q)}{D^j(p,q)}$$

and

$$D^j_j(p,q) = -\mathbb{E}_{\theta,\epsilon^j}\Big[\sum_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} \nu_g\Big(\triangle U_{jk}(\theta)\Big)\times\prod_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} G\Big(\triangle U_{jk}(\theta)\Big)\Big],$$

where $\nu_g(\cdot)\equiv g(\cdot)/G(\cdot)$ and $\triangle U_{jk}(\theta)\equiv U(y,q^j,p^j,\theta)-U(y,q^k,p^k,\theta)$.

The first term in the first-order condition in (1) is the Learner index that measures the intensity of competition. The greater the price elasticity of demand, the more intense the competition faced by firm $j$, since a price hike implies losing a large number of customers. The higher the elasticity, the smaller the markup as a share of the price.

Because the demand is log-supermodular in $(p^j,q^j)$ and log-submodular in $(p^j,q^k)$, the larger the firm $j$'s quality and the smaller the firm $k$'s quality, the less intense the competition faced by firm $j$. This happens because the customers who leave firm $j$ when it raises its price are not randomly selected among the customers patronized by provider

$j$. Customers with a higher willingness to pay for firm $j$'s quality are less sensitive to price changes. These are high valuation customers when provider $j$'s quality is higher than the competitors'.

The main difference between a public and a private provider, which is captured by the second term in equation (1), is that the former, ceteris paribus, is less concerned with profit margins. Thus, it chooses a lower price than an identical private provider that offers the same quality when each firm faces the same number of competitors of each type.

**Proposition 2.** *For any $(q, v, g, \beta) \in \Re_+^{n+N+2} \times [0, 1]$, there is a unique pure strategy Nash equilibrium in the pricing sub-game.*

Existence and uniqueness follow from log-concavity, log-supermodularity, constant marginal costs, and the fact that any provider's demand depends only on price differences, not on price levels.[11] Because firms' best responses are increasing, since demands are log-supermodular in $p$, there are the lowest and highest equilibria. Because demands are log-concave in price, i.e., the price semi-elasticity of demand falls with price, the slope of the best response for each firm is less than one, and therefore there is a unique fixed point.

Since the transformed game is supermodular in $p$ and has a unique equilibrium, it follows from Theorem 5 in Milgrom and Roberts (1990) that each player has only one serially undominated strategy. Because the set of serially undominated strategies is determined only by ordinal comparisons, the corresponding prices are also the unique serially undominated strategies in the original game. Hence, the original game has a unique equilibrium. This is a dominance-solvable and globally stable solution under any adaptive learning rule satisfying assumption A6 in Milgrom and Roberts (1990).

### 4.3 Comparative Statics

The next result provides comparative statics regarding $(g, v, \beta)$. This readily follows from the fact that the game is log-supermodular in $(p, -g, -v, \beta)$, private providers' best responses decrease with the voucher since $D_j^j < 0$, and public providers' best responses are

[11] This result holds when costs are convex.

independent of the voucher. The opposite is true for public providers' subsidies. Public providers' best responses are non-decreasing in $\beta$, and private providers' are independent of it.

**Proposition 3.** *For any $(q, v, g, \beta) \in \Re_{+}^{n+N+2} \times [0, 1]$, the equilibrium price profile $p(q)$ is non-increasing with $(g, v)$ and non-decreasing with $\beta$.*

This proposition shows that, holding everything else constant, equilibrium prices fall with the voucher. This occurs because an increase in the voucher raises private providers' markups, making a higher price less profitable as demand is negatively sloped. Public providers' best responses remain unaltered, and prices are strategic complements. The same happens when the per-customer public sector subsidy rises. The following corollary shows that this holds only if public providers' prices are strategically chosen and positive.

**Corollary 1** (Fixed Public Providers Prices)**.** *Suppose that a public agency sets the price of public providers to zero or $(g, \beta)$ is such that $p^j(q) = 0$ for all $j \in \mathcal{J}^0$. Then, for any $(q, v, g, \beta) \in \Re_{+}^{n+N+2} \times [0, 1]$ such that $p^j(q) > 0$ for all $j \in \mathcal{J}$, an increase in public providers' subsidy $g$ does not lower private providers' prices.*

Increasing the weight given to the markup $\beta$ results in higher prices. This happens because, in this case, public providers value profits more and prices are strategic complements.

The effect of an increase in provider $j$'s quality on provider $j$'s equilibrium price is ambiguous since the profit gain from raising the price might not increase with $q$, i.e., profits are not supermodular in $(p, q)$. Namely, firm $j$'s best response increases with $q_h$ whenever

$$\frac{\partial \log \Pi^j(p, q; \beta^j)}{\partial p^j q^h} = \left( c^j_{q^h}(q^j) \Big( \frac{D^j_j(p, q)}{D^j(p, q)} \Big)^2 + \frac{1}{p^j} \frac{\partial \eta^j(p, q)}{\partial q^h} \right) \Big|_{p = p(q)} > 0.$$

On the one hand, an increase in $q^j$ increases the marginal cost of production, which, ceteris paribus, raises firm $j$'s best response, and therefore, prices since they are strategic complements. On the other hand, it changes the price elasticity of demand. Holding prices constant, firm $j$'s price elasticity of demand increases with $q^j$ since $D^j_{jq^j} D^j - D^j_j D^j_{q^j} > 0$, since an increase in provider $j$'s quality increases customers' willingness to pay for provider $j$'s good. Thus, firm $j$'s price best response rises with $q^j$, which pushes prices up due to the strategic complementarity of prices. On the other hand, an increase in $q^j$, holding prices constant, decreases competitors' price elasticity of demand since $D^k_{kq^j} D^k - D^k_k D^k_{q^j} < 0$ and, also, provider $k$'s marginal cost is independent of $q^j$. Therefore, firm $k$'s best response falls with $q^j$, since an increase in provider $k$'s quality decreases customers' willingness to pay for provider $j$'s good. This force pushes prices down due to strategic complementarity. Thus, prices may rise or fall with an increase in $q^j$.

**Remark 2.** *A consequence of adding horizontal differentiation to a model of vertical differentiation with linear preferences is to break the positive relationship between prices and qualities. When there is both vertical and horizontal differentiation, the ranking of providers in terms of indirect utility differs across individuals, despite everyone preferring higher quality.*

To find a sufficient condition for $p^j(q)$ to increase with $q^j$, we use the Implicit Function theorem and the properties of $B_0$-matrices.[12] A $B_0$-matrix is one in which the mean of each row is positive and greater than the maximum between zero and each off-diagonal element in the same row. This always hold for a diagonally dominant matrix and, thereby, the standard dominant diagonal condition is a specific case of $B_0$ matrices. $B_0$ matrices have strictly positive diagonal and positive determinants, and their principal submatrices are all $B_0$-matrices, which means positive determinants too. Another valuable property is that the sum of the cofactors in each row is positive. Certainly, the same conclusions apply if the transpose of a matrix is a $B_0$-matrix, since the determinant of a matrix equals the determinant of its transpose.

Let $H(p, q; \beta)$ be the Hessian with respect to $p$ with entries $\left(\frac{\partial^2 \log \Pi^j(p,q;\beta^j)}{\partial p^j \partial p^h}\right)_{h,j \in \mathcal{J} \cup \mathcal{J}^0}$ and $H_q(p, q; \beta)$ be the matrix with entries $\left(\frac{\partial^2 \log \Pi^j(p,q;\beta^j}{\partial p^j \partial q^h}\right)_{h,j \in \mathcal{J} \cup \mathcal{J}^0}$. The matrix $H$ is a $B_0$-matrix when for all $j \in \mathcal{J} \cup \mathcal{J}^0$,

$$\sum_{h \in \mathcal{J} \cup \mathcal{J}^0} \frac{\partial^2 \log \Pi^j(p, q; \beta^j)}{\partial p^j \partial p^h} \leq (n + N) \min_{h \in \mathcal{J} \cup \mathcal{J}^0 \setminus j} \left\{0, \frac{\partial^2 \log \Pi^j(p, q; \beta^j)}{\partial p^j \partial p^h}\right\}.$$

[12]See, Peña (2001) and Christensen (2018), who uses $B_0$ matrices to derive comparative statics.

and the matrix $-H^T$ is a $B_0$-matrix when for all $j \in \mathcal{J} \cup \mathcal{J}^0$,

$$\sum_{h \in \mathcal{J} \cup \mathcal{J}^0} \frac{\partial^2 \log \Pi^h(p, q; \beta^j)}{\partial p^h \partial p^j} \leq (n+N) \min_{h \in \mathcal{J} \cup \mathcal{J}^0 \setminus j} \Big\{ 0, \frac{\partial^2 \log \Pi^h(p, q; \beta^j)}{\partial p^h \partial p^j} \Big\}.$$

Because profits are log-supermodular in $p$, $-H$ is $B_0$-matrix whenever

$$\left[ -\Big(\frac{D^j_j}{D}\Big)^2 + \frac{D^j D^j_{jj} - 2(D^j_j)^2}{(D^j)^2} + \sum_{h \neq j} \frac{D^j D^j_{jh} - D^j_j D^j_h}{(D^j)^2} \right]\Big|_{p=p(q)} \leq 0$$

and $-H^T$ is $B_0$-matrix whenever

$$\left[ -\Big(\frac{D^j_j}{D}\Big)^2 + \frac{D^j D^j_{jj} - 2(D^j_j)^2}{(D^j)^2} + \sum_{h \neq j} \frac{D^h D^h_{hj} - D^h_h D^h_j}{(D^h)^2} \right]\Big|_{p=p(q)} \leq 0.$$

Log-concavity of $D^j$ implies the first term is negative, and log-supermodularity in $p$ implies that the second term is positive. In this case, the condition turns out to be identical to the standard dominant diagonal condition.[13] Intuitively, it says that the impact of price $p^j$ on the firm's marginal profits is more important than the aggregated impact of competitors' prices on its marginal profits.

Because $\Pi^j$ is log-submodular in $(p^j, q^h)$ for any $j, h$ with $j \neq h$, the matrix $H_q$ is mean-positive dominant whenever

$$\left[ \Big(\frac{D^j_j}{D}\Big)^2 c'(q^j) + \frac{D^j D^j_{jq^j} - D^j_j D^j_{q^j}}{D^j_j} + \sum_{h \neq j} \frac{D^h D^h_{hq^j} - D^h_h D^h_{q^j}}{(D^h)^2} \right]\Big|_{p=p(q)} \geq 0.$$

Because demand $D^j$ falls with $p^j$ and is log-supermodular $(p^j, q^j)$, the first term is positive. The second term is negative because $D^j$ is log-submodular $(p^h, q^j)$. This condition establishes that an increase in its own quality raises the benefit of increasing the price more than the aggregated loss from raising the price when competitors increase their quality levels. The more sensitive the marginal cost to the quality provided, the more likely this condition is to be satisfied.

[13] This is a particular case of $-H$ being $B_0$ matrix. For details, see Christensen (2018).

**Proposition 4.** *Suppose that* $(q, v, g, \beta) \in \Re_{+}^{n+N+2} \times [0,1]$ *are such that* $p(q) > 0$,[14] $-H^T(p, q; \beta)$ *is a* $B_0$*-matrix, and* $-H_q(p, q; \beta)$ *is mean positive dominant in element j. Then, the equilibrium price* $p^j(q)$ *is non-decreasing in* $q^j$ *and* $p^h(q)$ *may either rise or fall with* $q^j$.

Because of quasi-linear consumer preferences, a price reduction would have the same effect on demand as a quality increase if $\theta$ were to be a fixed parameter since $D^j(p^j, p^{-j}; q^j + \delta, q^{-j}) = D^j(p^j - \theta\delta, p^{-j}; q)$. However, because different customers have different valuations, an increase in quality induces customers with higher quality valuations to switch. This makes the comparative statics concerning $q$ different from those that emerge from an exogenous decrease in the price, despite the utility differences depending on both quality and price. It follows from this and the increasing price elasticity of demand with respect to quality and the falling price elasticity with respect to competitors' quality that an unambiguous comparative statics requires restricting the size of the interaction effects relative to that of the direct impact.

The $B_0$-matrix assumption implies that the partial effect of an increase in $q^j$ on firm $j$'s price semi-elasticity of demand is larger than the effect on firm $j$'s competitors' price semi-elasticity of demand. Thus, the increase in firm $j$'s best response more than compensates for the decrease in competitors' best responses. When firms are symmetric and offer the same quality, this holds. However, when firms supply different quality levels, the larger the difference in quality, the greater the differences in the impact of private provider $j$'s quality on its price elasticity of demand and on competitors' price elasticity of demand. Thus, we cannot even be sure that higher-quality providers set higher prices in equilibrium. In addition, public providers place a positive weight on demand, making their sensitivity to their own and competitors' quality different from that of private providers.

[14] For any vector $x$, the notation $x > 0$ means that each component is strictly positive.

### 4.4 Exclusive Markets and Symmetric Mixed Markets

#### 4.4.1 Exclusive Market with Symmetric Firms

Let's assume that firms are symmetric: i.e., $c^j(\cdot) = c(\cdot),\ \forall\ j \in \mathcal{J} \cup \mathcal{J}^0$ and the market is supplied only by private or by public providers. The key message from this subsection is that in the absence of competition between public and private providers and symmetry, the selection in quality (cream-skimming) effects are equalized across firms and, therefore, prices are monotone in quality.[15]

It readily follows from the first-order condition in equation (1) for $q^j = q$ for all $j$ that the symmetric equilibrium price is given by

$$p(q) = \max\left\{0, c(q) - s - \frac{1-\beta^j}{\beta^j} + \frac{1}{m}\underbrace{\frac{1}{\int_{\underline{\epsilon}}^{\bar{\epsilon}} g(\epsilon)dG(\epsilon)^{m-1}}}_{\text{competiton effect}}\right\}. \tag{2}$$

where $(m,s) = (N,g)$ and $\beta^j = \beta$ when providers are public and $(m,s) = (n,v)$ and $\beta^j = 1$ when they are private.

The numerator in equation (2) is the equilibrium demand. The denominator is the slope of the demand. This term is the competition effect, i.e., the density of a firm's marginal customers, those who are indifferent between the corresponding firm and the best alternative firm for them, multiplied by the loss from a lower probability of being patronized. It is easy to see that the pass-through from subsidies to prices is -1, and from quality to prices is $p'(q) = c'(q)$.

Let $q^0$ be the public sector quality and $q^1$ the private sector quality when firms are symmetric within a sector. Then, when $v = g$ and $q = q^0 = q^1$ and $p^0(q) > 0$, equation (2) implies that the price difference between private and public providers is $-\frac{1-\beta}{\beta}$. Thus, the equilibrium price in a purely publicly-provided market is lower than the equilibrium price in a strictly private market for $g \leq v - (1-\beta)/\beta$ and $q^0 \leq q^1$, and the difference rises with $\beta$ since a larger $\beta$ implies that public providers care relatively more about profits

[15] This is also driven by the fact that we consider fully covered markets.

than market share. In the limit, when $\beta$ goes to 1, the price charged by pubic providers is identical to that set by private providers when $v = g$, $n = N$, and $q^1 = q^0$, whereas when $\beta$ goes to 0, public providers' price goes to zero.

**Remark 3.** *Partial Coverage. It can be shown that, with partial coverage, the optimal price is the unique solution to the following fixed-point equation.*[16]

$$p = \max\left\{0, c(q) - s - \frac{1-\beta^j}{\beta^j} + \frac{1}{m}\frac{\mathbb{E}_\theta[1 - F^m(\phi(p,q))]}{\underbrace{\mathbb{E}_\theta[G^{m-1}(\phi(p,q))g(\phi(p,q))]}_{\text{exclusion effect}} + \underbrace{\mathbb{E}_\theta\Big[\int_{\phi(p,q)}^{\bar{\epsilon}} g(\epsilon)dG(\epsilon)^{m-1}\Big]}_{\text{competition effect}}}\right\}. \tag{3}$$

*where* $\phi(p,q) \equiv \max\{\underline{\epsilon}, p - \theta q - y\}$.

*The numerator in equation* (3) *is the equilibrium demand. The denominator is the slope of the demand, taking into account the outside option. This has two terms: (i) the market exclusion effect when the valuations for all other firms are below* $\phi(p,q)$*, which occurs with probability* $pG(\phi(p,q))^{m-1}$*, firm* $j$ *acts as a monopoly. Increasing its price* $p$ *by* $\epsilon$ *will exclude* $\epsilon g(\phi(p,q))$ *individuals from the market; and (ii) the competition effect (up to the adjustment that the marginal consumer's valuation for the good is given by* $\phi(p,q)$*). This term is equivalent to that in equation* (2) *and has the same interpretation.*

*In this case, quality has an impact on the consumer's decision between patronizing a firm and taking his outside option. Thus, higher quality results in stratification (cream-skimming), i.e., those who value quality more participate in the market. The remainder takes the outside option with payoff* $y$*.*

### 4.4.2 Mixed Markets with Symmetric-by-Sector Firms

The case in which all firms are symmetric within each sector serves as a natural benchmark, yielding clearer analytic results and intuition. Let's assume providers have the same marginal cost and quality level within each sector. The quality profile is given by $q = (q^1, q^0)$ and the price profile $p = (p^1, p^0)$.

[16] See Zhou (2017) for a formal proof.

Let $\eta^1 < 0$ be private providers' semi elasticity, $\eta_1^{m1} = \sum_{i \in \mathcal{J}^1} \frac{\partial}{\partial p^i} \left( \frac{D_j^j}{D^j} \right)$ for $j \in \mathcal{J}^1$ be the derivative of the private-market demand semi-elasticity with respect to private providers' prices. This is positive when demand is convex in prices. Let's define the analog terms for public providers by $\eta^0$ and $\eta_0^{m0}$, respectively, and the cross terms $\eta_0^{m1} > 0$ and $\eta_1^{m0} > 0$, where the sign follows from the log-supermodularity of demand in $(p^1, q^1)$ and $(p^0, q^0)$, respectively. Totally differentiating the first-order condition for prices and imposing symmetry, we can show the following.

**Corollary 4.** *Suppose firms are symmetric within each sector. Let $(p^1(q), p^0(q))$ be the Nash-equilibrium prices for private and public providers, respectively. Then, for any $(q, v, g, \beta) \in \Re_+^{n+N+2} \times [0, 1]$, such that $p(q) >> 0$. Then:*

*i) If $n = N$, $q^1 = q^0 = \hat{q}$, and $g = v - \frac{1-\beta}{\beta}$, then $p^1(\hat{q}, \hat{q}) = p^0(\hat{q}, \hat{q})$. Furthermore, if $q^1 > q^0 = \hat{q}$, $p^1(q^1, \hat{q}) > p^0(q^1, \hat{q})$ for all $g \geq v - \frac{1-\beta}{\beta}$.*

*ii) The equilibrium rate of voucher pass-through equals:*

$$p_v^1(q) = -\frac{(-(\eta^0)^2 + \eta_0^{m0})(\eta^1)^2}{(-(\eta^1)^2 + \eta_1^{m1})(-(\eta^0)^2 + \eta_0^{m0}) - \eta_0^{m1}\eta_1^{m0}} < 0$$
$$p_v^0(q) = \frac{\eta_1^{m0}(\eta^1)^2}{(-(\eta^1)^2 + \eta_1^{m1})(-(\eta^0)^2 + \eta_0^{m0}) - \eta_0^{m1}\eta_1^{m0}} < 0.$$

*iii) Suppose that $-H(p, q; \beta)$ is a $B_0$-matrix in $p$, then $p_v^1 \leq p_v^0$ and $p_g^1 \geq p_g^0$.*

The first part says that if qualities and competition intensity across sectors are the same, then prices are the same when the voucher is lower than the per-student subsidy in an amount $(1-\beta)/\beta$ because of public providers' mandate to be concerned with market share, which induces them to choose lower prices to make their sector more attractive. Furthermore, suppose private providers' quality is higher. In that case, private providers' prices are higher than public providers' when the voucher is equal to or smaller than the per-student subsidy plus $(1-\beta)/\beta$ since prices fall with the voucher.

This follows from the fact that profits are log-concave in their own price, log-supermodular in competitors' prices, and that firms' best responses increase with their own quality and decrease with their competitors' quality. Thus, whenever the quality of private providers improves, they tend to raise prices, while public providers either raise them by a smaller amount or lower them.

The second part establishes that if firms are symmetric within a sector and the Hessian satisfies the regularity condition posed in the proposition, then the pass-through from the voucher to private providers' price is larger than to public providers'. Thus, holding quality constant, a large voucher decreases prices and, eventually, could make private providers less expensive than public providers. This happens because an increase in the voucher, holding prices constant, increases private providers' markups and keeps public providers' unchanged, and prices are strategic complements.

**Corollary 5.** *Suppose that* $(q, v, g, \beta) \in \Re^4_+ \times [0,1]$ *are such that* $(p^0(q), p^1(q)) > 0$*. If* $-H^T(p, q; \beta)$ *is a* $B_0$*-matrix in* $p$ *and* $-H_q(p, q; \beta)$ *is mean-positive dominant in* $q^1$*, then* $p^1_{q^1} > 0$ *and* $p^1_{q^1} \geq p^0_{q^1}$*, whereas if it is mean-positive dominant in* $q^0$*, then* $p^0_{q^0} > 0$ *and* $p^0_{q^0} \geq p^1_{q^0}$*.*

The corollary establishes that an increase in private providers' quality results in higher private providers' prices and higher relative prices. The opposite occurs with an increase in the quality of public providers. This happens because private providers' best responses increase with their quality, due to higher marginal costs, and their demand becomes less price-elastic, since customers who patronize public providers are now willing to pay more for a private provider. In contrast, the marginal cost of public providers remains unchanged, and their demand becomes more elastic. Similarly, for an increase in the quality of public providers.

In Table 1, we present numerical comparative statics regarding increases in public and private providers' quality when the equilibrium is symmetric within each sector. The first row corresponds to $v = 6$ and the next to $v = 7$.

The relationship between prices and vouchers is negative, as predicted. In every case, the pass-through from the voucher to private providers' prices is higher than -1, and it is even higher for public providers' prices. Thus, $p^1_v - p^0_v < 0$.

The relationship between public providers' quality and equilibrium prices is monotonic. Prices increase in its quality and decrease in competitors' quality. The pass-trough from $q^1$ to prices is such that $p^1_{q^1} - p^0_{q^1} > 0$ and from $q^0$ is such that $p^1_{q^0} - p^0_{q^0} < 0$.

Prices are higher in the public sector when quality is much higher than in the private sector, and vice versa.

| $(q^0, q^1)$ | 16, 18 | 17, 18 | 18, 18 | 19, 18 | 20,18 | 18, 19 | 18, 20 |
|---|---|---|---|---|---|---|---|
| $(p^0, p^1)\|_{v=6}$ | 24.36, 33.65 | 28.24, 33.14 | 32.44, 32.44 | 36.82, 31.76 | 41.20, 31.25 | 31.76, 36.82 | 31.25, 41.20 |
| $(p^0, p^1)\|_{v=7}$ | 24.24, 32.71 | 28.12, 32.22 | 32.32, 31.55 | 36.72, 30.88 | 41.14, 30.35 | 31.65, 35.90 | 31.13, 40.26 |

**Table 1.** Comparative statics regarding prices concerning quality
$v \in \{6, 7\}$, $g = v - (1-\beta)/\beta$, $c(q) = 0.1q^2$, $\beta = 0.8$, $n = N = 6$, $\epsilon \sim \mathcal{N}(0, 10; -30, 30)$ and $\theta \sim \mathcal{N}(10, 2; 1, 100)$.

# 5 The Quality Sub-Game

## 5.1 The Equilibrium

When we substitute the equilibrium price into the profit function, provider $j$'s profit maximization problem becomes:

$$\max_{q^j \in Q} \{\Pi^j(p(q), q; \beta^j) - C^j(q^j)\}.$$

Let's denote $D^j(p(q), q)$ by $D^j(q)$. Due to the envelope Theorem, provider $j$'s first-order condition for $q^j$ is as follows

$$(\beta^j(p^j(q) + s^j - c^j(q^j)) + 1 - \beta^j) D^j_{q^j}(q)|_{p^j=k} - \beta^j c^{j'}(q^j) D^j(q) - C^{j'}(q^j) = 0. \tag{4}$$

where $s^j \in \{v, g\}$ and the partial derivative of provider $j$'s demand concerning its quality when its price is held constant is equal to

$$D^j_{q^j}(q)\big|_{p^j=c} = \mathbb{E}_{\theta,\epsilon}\left[\theta \sum_{k \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}} \nu_g\Big(\triangle U_{jk}(\theta)\Big) \times \prod_{k \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}} G\Big(\triangle U_{jk}(\theta) + \epsilon^j\Big)\right] + \sum_{h \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}} D^j_h(p, q) p^h_{q^j}(q).$$

The first-order condition for $q^j$ in equation (4) can be explained by three different effects: the income gained from the business-stealing effect, the income gained/lost from the strategic-commitment effect, and the loss due to a higher cost to provider higher quality.

The business-stealing effect corresponds to new customers switching to provider $j$ because of higher quality when competitors' prices are constant. They come from both competing private and public providers.

The strategic-commitment effect measures the customers lost or gained due to changes in the equilibrium prices of competing providers induced by provider $j$'s quality. When a competitor raises its price in response to higher quality, the strategic-commitment effect implies a gain in customers, since they would be paying higher prices if they kept their choice of provider unchanged and lower quality relative to the new situation. By contrast, when competitors' prices fall, firm $j$ loses customers because competitors become more attractive. This hits harder on public providers since their objective functions place a positive weight on their market share. As shown in Proposition 4, firm $j$'s competitor prices may either rise or fall with quality $q^j$.

The customers drawn to provider $j$ by either effect are not randomly drawn from competing providers; they are the ones who value quality the most, whereas those who prefer competing providers have the lowest valuation for the increase in quality among all those who have not been choosing provider $j$ before quality improves.

Finally, the cost effect corresponds to the increase in the marginal cost of serving a customer with quality times the number of customers plus the direct cost of improving quality. The larger the number of customers patronizing provider $j$, the higher the total costs. Furthermore, the lower the $\beta^j$, the less provider $j$ cares about the marginal income minus marginal cost of quality and more about market share.

From now on, we assume the following:

**Assumption 1.** *For all $\beta^j \in [0,1]$, $\Pi^j(p(q), q; \beta^j) - C^j(q^j)$ is quasi-concave in $q^j$.*

A sufficient condition for this to hold is that $D^j(q)$ is log-concave in $q^j$ and $p^j(q) - c(q^j)$ is concave in $q^j$.

The next result follows from the Debreu-Glicksberg-Fan's Theorem.

**Proposition 5.** *Suppose that Assumption 1 hold. Then, there exists a sub-game perfect equilibrium* $(q(v,g), p(v,g)) \in \Re_{+}^{2(N+n)}$*. If firms are symmetric,* $\beta = 1$*,* $N = n$*, and* $v = g$ *the equilibrium is symmetric, whereas if* $\beta^j < 1$*, there is no symmetric equilibrium.*

## 5.2 Quality and Vouchers When Markets Are Exclusive

In this sub-section, we compare the equilibrium when there are only symmetric private providers with that in which there are only symmetric public providers.

It readily follows from the first-order condition in equation (4) and the equilibrium condition for prices in equation (1) that the quality is the solution to[17]

$$\frac{\beta^j}{m}\bar{\theta} - C'(q) = 0, \tag{5}$$

where $(m,s) = (N,g)$ and $\beta^j = \beta$ when providers are public and $(m,s) = (n,v)$ and $\beta^j = 1$ when they are private.

It readily follows that the equilibrium quality, denoted by $q^1(s)$, is independent of $s$, raises with $\bar{\theta}$, falls with $m$, and raises with $\beta$.

The equilibrium quality is independent of the voucher for two reasons: first, the pass-through from the voucher to prices is -1, implying that vouchers do not change markups; and second, demand is independent of the voucher. These equilibrium features happen because customers' indirect utility is linear in income (prices), the marginal utility of quality is independent of income, and there is full coverage. Hence, the impact of prices on the marginal customer is independent of the quality. This is no longer the case when there is partial coverage or competition between private and public providers.

Quality rises with $\beta^j$ because providers care more about profits and less about market share; prices increase with $\beta^j$, and therefore the markup, holding quality constant, is higher.

The following result is deduced from equations (2) and (5).

**Proposition 6.** *Suppose within-sector-firms are symmetric and* $(v,g,\beta) \in \Re_{+}^{2} \times [0,1]$ *is such*

[17] The objective function is strictly concave in $q^j$ and, thereby, a sub-game equilibrium in which quality is positive exists and is unique.

*that $p(q) > 0$. Then, privately-provided quality when customers are served exclusively by private providers is larger than publicly-provided quality when customers are served exclusively by public providers for all $N \geq n\beta$. Whenever $n \leq N$, private providers' price exceeds public providers' price whenever $c(q^1) - c(q^0) \geq v - g - \frac{1-\beta}{\beta}$.*

Because public providers focus on profits and market share, they charge lower prices and offer lower-quality services. The quality they offer decreases as their market share weight $1-\beta$ increases. This happens because public providers' prices rise with $\beta$, which means ceteris-paribus a larger profit margin, and higher quality yields a higher demand. Therefore, the marginal return to quality rises with $\beta$.

Under the full-coverage assumption, neither vouchers nor per-customer subsidies affect quality. Any effect on quality results from competition between public and private providers, and having different objective functions.

**Remark 6.** *Partial Coverage. When there is partial coverage, the first-order condition for the symmetric equilibrium quality, after substituting into the first-order condition for quality, the equilibrium condition for prices, is*

$$\beta^j \big(P_q(p,q) + P(p,q) D_q(p,q)\big)|_{p=p(q)} - C'(q) = 0$$

*where*

$$P(p,q) \equiv \frac{1}{m} \frac{\mathbb{E}_\theta[1 - F^m(\phi(p,q))]}{\underbrace{\mathbb{E}_\theta[G^{m-1}(\phi(p,q))g(\phi(p,q))]}_{\text{exclusion effect}} + \underbrace{\mathbb{E}_\theta\Big[\int_{\phi(p,q)}^{\bar{\epsilon}} g(\epsilon) dG(\epsilon)^{m-1}\Big]}_{\text{competition effect}}}.$$

*and $\phi(p,q) \equiv \max\{\underline{\epsilon}, p - \theta q - y\}$.*

*Thus, if the symmetric equilibrium results in partial coverage, the optimal quality depends on the voucher/subsidy. Because the function $P(p,q)$ is decreasing in $\phi(p,q)$, due to log-concavity of $G(\cdot)$, and this rises with $p$, $P(p(q),q)$ rises with the voucher/subsidy. Because $P_{pq} \lesseqgtr 0$, the impact of higher voucher/subsidy on quality is ambiguous.*

### 5.3 Comparative Statics: Quality and Vouchers

Let $H(q;\beta)$ be the Jacobian of the first-order conditions for qualities evaluated at the equilibrium $(p(v,g),q(v,g))$ (hereinafter, the Hessian). By the Implicit Function theorem, we have that for all $j \in \{0,1\}$,

$$q^j_v(v,g) = -\frac{\sum_{i\in\mathcal{J}\cup\mathcal{J}^0}\Pi^i_{q^i v}H^{ij}}{\sum_{i\in\mathcal{J}\cup\mathcal{J}^0}\Pi^i_{q^i q^j}H^{ij}}\Big|_{(q(v,g),p(v,g))}. \tag{6}$$

where $H^{ij}$ is the $ij$ co-factor from the Hessian of the second-order conditions for quality, denoted by $H(q;\beta)$.

An increase in the voucher raises provider $j$'s best response when

$$\Pi^j_{q^j v} = \beta^j\left[\left(\frac{\partial p^j(q)}{\partial v}+1\right)D^j_{q^j}(q)|_{p^j=k} - c'(q^j)D^j_v(q) - \frac{D^j(q)}{D^j_j(q)}D^j_{q^j v}(q)|_{p^j=k}\right] \tag{7}$$

where

$$D^j_v(q) = \sum_{h\in\mathcal{J}\cup\mathcal{J}^0} D^j_h p^h_v(q) = \sum_{h\in\mathcal{J}\cup\mathcal{J}^0\setminus j} D^j_h(p^h_v(q)-p^j_v(q)),$$

$$D^j_{jv}(q) = \sum_{h\in\mathcal{J}\cup\mathcal{J}^0} D^j_{jh} p^h_v(q) = \sum_{h\in\mathcal{J}\cup\mathcal{J}^0\setminus j} D^j_{jh}(p^h_v(q)-p^j_v(q)),$$ [18]

and

$$D^j_{q^j v}(q)|_{p^j=k} = \underbrace{\sum_{h\in\mathcal{J}\cup\mathcal{J}^0} D^j_{q^j h}p^h_v(q)}_{\text{Change in the Business-Stealing Effect}} + \underbrace{\sum_{h\in\mathcal{J}\cup\mathcal{J}^0\setminus j} D^j_h p^h_{q^j v}(q) + \sum_{i\in\mathcal{J}\cup\mathcal{J}^0}\sum_{h\in\mathcal{J}\cup\mathcal{J}^0\setminus j} D^j_{hi}p^h_{q^j}(q)p^i_v(q)}_{\text{Change in the Strategic-Commitment Effect}}.$$

The first term inside the parentheses measures how the markup changes with the voucher, holding quality constant. Because for any $q$, prices fall with the voucher, the equilibrium markup could either increase or decrease with the voucher since $p^j_v(q) \lesseqgtr -1$. When the pass-through from the voucher to prices exceeds $-1$, the markup rises.

The sum of the second and third terms inside the parentheses measures how the difference between the marginal income, when the markup is held constant, and the marginal cost of serving customers changes with the voucher. This depends on: i) how the demand changes with the voucher when quality is held constant; and ii) how the business-stealing and strategic-commitment effects change with the voucher. The business-stealing effect increases with the voucher when demand falls with prices, as prices decrease with the voucher. This requires that the aggregated impact of the fall in competitors' prices more than compensate for the fall in provider $j$'s price. In contrast, the strategic-commitment effect does so when: (i) $p^j_{q^j}(q) \geq 0$ and $p^h_{q^j}(q) \leq 0$ for all $h \neq j$ since firm $j$'s demand is log-concave in $p^j$ and supermodular in $p$ and (ii) $\sum_{h \in \mathcal{J} \cup \mathcal{J}^0 \setminus j} D^j_h p^h_{q^j v}(q) \geq 0$. Because $D^j_h > 0$, this holds when the change in competitors' price response to provider $j$'s quality with a hike in the voucher is larger than the provider $j$'s price response to its quality. These are demanding conditions that are hard to satisfy and corroborate in models other than those with linear demands.

The third term in equation (7) is negative since demand is decreasing. Therefore, when demand is smaller, a higher marginal cost has a lower impact on total costs.

To better grasp the economics underlying the relationship between quality choices and vouchers, we will consider the case in which prices are regulated or fixed in both the public and private sectors. This case is also interesting in its own right since in many markets where competition between public and private providers occurs, prices are underregulated. This is common in education markets and less so in health care markets.[19]

When prices are fixed, there is no strategic-commitment effect, and thereby, $\Pi^j_{q^j s} = \beta^j D^j_{q^j} > 0$. This means that an increase in the voucher increases private providers' best responses, since markups rise, while public providers' best responses remain unaltered. In contrast, when the public-sector per-customer subsidy rises, public providers' best responses increase, and private providers' best responses do not change.

[19]This is the case in the Chilean education and health care markets.

Qualities are strategic complements when for all $j, k$ with $j \neq k$,

$$D^j_{q^j q^h}(q)\big|_{p=c} = -\mathbb{E}_{\theta,\epsilon}\Bigg[\theta^2\Bigg(\nu'_g\Big(\triangle U_{jk}(\theta)\Big) + \nu_g\Big(\triangle U_{jk}(\theta)\Big) \times \sum_{k \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}} \nu_g\Big(\triangle U_{jk}(\theta)\Big)\Bigg) \times \prod_{k \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}} G\Big(\triangle U_{jk}(\theta) + \epsilon^j\Big)\Bigg] \geq 0,$$

**Proposition 7.** *Suppose prices are fixed.*

*i) Suppose that* $D^j_{q^j q^h}(q)\big|_{p=c} \geq 0$ *for all* $i, h$, $i \neq h$ *and for all* $q$. *Then, a lowest* $q_L(v, g)$ *and a largest equilibrium* $q_H(v, g)$ *exist. In addition* $q_L(v, g)$ *and* $q_H(v, g)$ *are non-decreasing in* $(v, g)$.

*ii) Suppose that* $D^j_{q^j q^h}(q)\big|_{p=c} < 0$ *for all* $i, h$, $i \neq h$ *and* $H^T(q; \beta)$ *is a* $B_0$*-matrix at* $q = q(v, g)$. *Then, for* $j \in \mathcal{J}$, $q^j(v, g)$ *rises with* $v$ *whenever*

$$D^j_{q^j}(q)\big|_{p=c} + \sum_{k \in \mathcal{J} \setminus \{j\}} D^k_{q^k}(q)\big|_{p=c} \geq n \max_{k \in \mathcal{J} \setminus \{j\}} \{D^k_{q^k}(q)\big|_{p=c}\}.$$

*If* $D^j_{q^j}(q)$ *is identical for all* $j \in \mathcal{J}$, *then for* $q^j(v, g)$ *rises with* $v$ *for all* $j \in \mathcal{J}$ *and falls with* $v$ *for all* $j \in \mathcal{J}^0$.

*iii) Suppose that* $D^j_{q^j q^h}(q)\big|_{p=c} < 0$ *for all* $i, h$, $i \neq h$ *and* $H^T(q; \beta)$ *is a* $B_0$*-matrix at* $q = q(v, g)$. *Then, for* $j \in \mathcal{J}^0$, $q^j(v, g)$ *rises with* $g$ *whenever*

$$D^j_{q^j}(q)\big|_{p=c} + \sum_{k \in \mathcal{J}^0 \setminus \{j\}} D^k_{q^k}(q)\big|_{p=c} \geq N \max_{k \in \mathcal{J}^0 \setminus \{j\}} \{D^k_{q^k}(q)\big|_{p=c}\}.$$

*If* $D^j_{q^j}(q)$ *is identical for all* $j \in \mathcal{J}^0$, *then for* $q^j(v, g)$ *rises with* $g$ *for all* $j \in \mathcal{J}^0$ *and falls with* $g$ *for all* $j \in \mathcal{J}$.

*iv) Suppose that* $H(q; \beta)$ *is a* $B_0$*-matrix at* $q = q(v, g)$. *Then,* $\sum_{j \in \mathcal{J} \cup \mathcal{J}^0} q^j_s \geq 0$ *for* $s \in \{v, g\}$.

When qualities are strategic complements, an increase in the voucher and/or the per-customer subsidy will result in higher quality. This happens because the voucher increases private providers' profit margins while leaving public providers' unaltered, making quality more profitable. Because of the complementarity, this induces competitor to raise their quality, which reinforces the initial increase. The same happens when the per-customer subsidy rises.

When qualities are strategic substitutes, the increase in the voucher raises private providers' best responses and holds public providers' one constant. Because private providers' qualities are strategic substitutes, an increase in quality by one provider induces the others to decrease quality, which, in turn, reinforces the partial effect. Quality rises for private providers whose partial effects dominate the interaction effects. When all private providers are identical, their quality increases as the voucher rises, while public providers' quality decreases. The opposite happens when the per-student subsidy rises.

The convexity of the demand curve determines whether qualities are strategic complements or substitutes. A sufficient condition for qualities to be substitutes is that $\prod_{k \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}} G\left(\triangle U_{jk}(\theta) + \epsilon^j\right)$ is convex in $\epsilon^j$ for all $j$. Whereas qualities are strategic complements, the opposite is true. This holds whenever $n + N$ is large enough. The opposite could hold when $n + N$ is small enough due to the log-concavity of $g$. This follows from the following lemma.

**Lemma 1.** *There is a threshold $m$ such that $\prod_{k \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}} G\left(\triangle U_{jk}(\theta) + \epsilon^j\right)$ is convex for all $n + N \geq m$ irrespective of the shape of $g(\cdot)$.*

Thus, when competition intensity, as measured by the number of providers, is sufficiently strong, quality increases for some providers while it falls for others.

When prices are not fixed, the analysis is more complex due to: i) the existence of the strategic-commitment effect, and ii) the fact that the pass-through from subsidies to prices is negative and, therefore, markups could be smaller or larger after the increase in the subsidies.

The following proposition is the counterpart of Proposition 7 when prices are endogenous and decrease with subsidies as shown in Proposition 3.

**Proposition 8.** *Suppose $(v, g, \beta) \in \Re^2_+ \times [0, 1]$ is such that $p(q) > 0$.*

*i) Suppose that $H(q;\beta)$ is a $B_0$-matrix in $q$.*[20] *If*

$$\min_{h\in\mathcal{J}\cup\mathcal{J}^0}\{\Pi^h_{q^h v}(p(q),q;\beta^h)\}\geq 0,$$

*then $\sum_{j\in\mathcal{J}\cup\mathcal{J}^0} q^j_v \geq 0$.*

*ii) Suppose that $H^T(q;\beta)$ is a $B_0$-matrix in $q$. If*

$$\sum_{h\in\mathcal{J}\cup\mathcal{J}^0}\Pi^h_{q^h v}(p(q),q;\beta^h)\geq (n+N)\max_{h\neq j}\{0,\Pi^h_{q^h v}(p(q),q;\beta^h)\},$$

*then $q^j_v \geq 0$.*

Aggregated quality increases with the voucher when the best responses regarding quality rise with the voucher, irrespective of whether qualities are complements or substitutes. When qualities are strategic complements, this is straightforward. In contrast, when they are substitutes, this requires that neither private nor public providers' best responses change by much with the voucher, since the drop in some providers' quality could be offset by an increase in others' quality, or vice versa. The $B_0$-matrix assumption limits the size of the interaction effects by imposing that the average impact in marginal returns exceeds the most significant interaction effect.

The quality of firm $j$ increases with $v$ whenever firm $j$'s best response increases with $v$, and the average increase in best responses is larger than the most significant increase in the best response of firm $j$'s competitors. Thus, again, the partial effect more than compensates for the interaction effects. This, together with the assumption that $H^T(q;\beta)$ is a $B_0$-matrix in $q$, ensures that the partial effect of $q^j$ on firm $j$'s best response outweighs the interaction effects from competitors' optimal responses to a larger $q^j$.

**Strategic Complements** In this case, as private providers become more aggressive, public providers respond by being more aggressive too. This feature provides competition

[20]Recall that this is less stringent than assuming dominance diagonal, the standard assumption in oligopoly comparative statics.

with vouchers, the best scenario for a positive impact on quality, as an increase in vouchers raises the quality of private providers. Qualities increase whenever private and public providers' best responses rise (weakly) with the voucher; that is, $\Pi^j_{q^j v} \geq 0$ because the increase in $q^j$ increases the marginal return to $q^k$ and vice versa.

**Strategic Substitutes** When a private provider increases its quality, competitors' best responses fall, and they wish to offer a lower quality. For the voucher to be the tide that lifts all boats, it must be that the voucher increases every provider's best response, i.e., $\Pi^j_{q^j v} > 0$ for all $j$, and the difference in both the partial and the interaction effects cannot be too large, so as the increase in competitors' quality overcomes the partial effect of each provider. No tide lifts all boats when any competitor's best response falls, i.e., $\Pi^j_{q^j v} \leq 0$ for some $j$. In the former case, the best responses of all providers increase similarly. This, together with the fact that the transpose of the Hessian is $B_0$-matrix, ensures that the quality level offered by all providers increases. In the latter case, the tide that lifts all boats does not exist because providers are less aggressive, and a larger voucher makes some providers even less aggressive; their incentive to lower quality is therefore more substantial.

For private providers' best responses to increase with the voucher, the change in the sum of the business-stealing and strategic-commitment effect has to more than compensate for the increase in total marginal costs and the decrease in markups when the pass-through from the voucher to prices is higher than $-1$.

# 6 Empirical Evidence

## 6.1 Educational Markets

This subsection summarizes empirical evidence on competition between private and public education providers and the effects of vouchers worldwide, emphasizing heterogeneous outcomes and contextual dependence (see, for instance, Urquiola (2016), Epple, Romano, and Urquiola (2017), and MacLeod and Urquiola (2019)). It provides a more detailed discussion of evidence from the United States, Chile, and Sweden, where substantial empirical work exists.

Randomized controlled trials, quasi-experiments, and cross-country analyses show that vouchers and private provision can increase enrollment and short-term attendance in contexts with weak public provision (Angrist, Bettinger, Bloom, King, and Kremer (2002); Andrabi, Das, and Khwaja (2011)). However, learning gains are uneven: some studies record modest test-score improvements, while others find negligible or no effects on learning once selection and peer effects are accounted for (Angrist et al. (2002); Barrera-Osorio, de Barros, Filmer, Martinez, Ripani, and Santibanez (2011); Andrabi et al. (2011)). Meta-analyses emphasize that accountability (through testing and inspections), adequate voucher size, and accreditation are critical to translating increased access into learning improvements (Patrinos, Barrera-Osorio, and Guaqueta (2009); Vegas and Petrow (2008)).

Vouchers and private schools in low-income settings often offer perceived advantages—discipline, responsiveness, flexible hours—but face challenges in teacher qualifications, curriculum coverage, and regulatory oversight, producing heterogeneous quality (Andrabi et al. (2011); Barrera-Osorio et al. (2011)). Distributional impacts vary: poorly designed programs can increase stratification; targeted vouchers or weighted funding that favor disadvantaged students tend to mitigate—but not eliminate—such risks (Ladd and Fiske (2009); Hsieh and Urquiola (2006)).

The U.S. literature is extensive and varied across program types. Charter schools, private school voucher programs, and inter-district choice policies have been evaluated using lottery-based, difference-in-differences, and regression discontinuity designs. The evidence suggests substantial heterogeneity across charter operators: while many charters underperform, some high-performing networks generate notable gains, particularly for low-income and minority students (Angrist, Pathak, and Walters, 2013, Walters, 2018, Gleason, Clark, Tuttle, Dwoyer, and Peterson, 2010). However, they are less likely to apply to them. Lottery-based studies of specific charter systems reveal positive impacts in some urban networks (e.g., Boston, New York) but negligible or negative effects elsewhere (Angrist et al., 2013, Abdulkadiroğlu, Pathak, and Walters, 2018). Research also highlights sorting: choice programs can reallocate higher-achieving or more-engaged students to certain providers, affecting peer composition and spillovers for remaining public schools (Hoxby, 2000, Ladd and Fiske, 2009).

Chile provides one of the most studied national voucher systems. Initial reforms in the 1980s introduced universal vouchers and encouraged private subsidized schools, producing a rapid expansion of private providers. Empirical analyses find mixed effects: some studies report improvements in school efficiency and responsiveness, while others document increased segregation by socioeconomic status and mixed effects on learning. Hsieh and Urquiola (2006) show that competition led to sorting and that average achievement gains were limited once selection is accounted for. Other work suggests that accountability measures, testing regimes, and centralized oversight moderate outcomes; when these are weak, voucher-driven competition can exacerbate inequality (see, for instance, Mizala and Romaguera (2000)). Cartagena and McIntosh (2019), using Chilean schools, find that an increase in the number of voucher schools in a local area, holding the number of public schools constant, does not improve performance at other schools in the same area; if anything, the relationship is slightly negative. In contrast, an increase in the number of public schools in an area, holding the number of voucher schools constant, raises test scores in other schools in the area. Neilson (2020) find that profit-maximizing voucher schools choose quality below the competitive level, and the higher the market power, the lower the quality. The quality markdown is greater in poorer areas, where households are estimated to be more price sensitive. As such, vouchers that are higher for poorer households have a greater positive effect on quality. Feigenberg, Yan, and Rivkin (2019) find that the reform resulted only in a small increase in resources and mobility across schools and little evidence of improvements driven by competition, but a closing of the parental education and income gaps, raising doubts that the program accounts for much of this convergence.

Longer-term studies of Chilean reforms reveal complex dynamics: the market entry of private schools increased parental choice and diversification, but also led to cream-skimming and stratification across neighborhoods and socioeconomic groups. Recent policy debates in Chile have focused on rebalancing autonomy, accountability, and equity, including adjustments to funding formulas and admissions rules to reduce segregation and improve system-wide outcomes.

Sweden's large-scale voucher reform in the 1990s opened public education to private, per-student-funded independent schools. Empirical assessments find mixed outcomes: Böhlmark and Lindahl (2015) document that vouchers increased school variety and had modest positive effects on student achievement for some cohorts, but segregation by parental income and immigrant status increased. Other studies highlight that competition spurred innovation and parental satisfaction in some municipalities but amplified disparities where municipal oversight and redistributive funding were limited (Fischer and Sebring-style comparisons omitted). The Swedish case underscores how generous choice policies without compensating redistributive mechanisms can raise stratification even if average achievement gains occur for some groups (Carnoy, 1998).

The empirical literature reveals no universal prescription: competition and vouchers can expand access and, in specific implementations, raise learning outcomes—especially for disadvantaged students when accompanied by accountabilsuggests an ambiguous relationship between competition and quality in the healthcare sector, with studies highlighting the complex interplay among.S., Chile, and Sweden illustrate the range of possible outcomes and reinforce the importance of careful program design, regulation, and monitoring in achieving desirable educational and distributional goals.

# 7 Competition between Private and Public Health Providers and the Role of Vouchers

The evidence points to an ambiguous relationship between competition and quality in the healthcare sector, with studies highlighting the complex interplay of patient characteristics, strategic hospital behavior, and contextual factors.

Tay (2003), for instance, analyzed individual-level Medicare data on heart attack patients and found that both quality and distance are crucial determinants of hospital choice. His research emphasized that the trade-off between quality and distance, as well as the valuation of different quality aspects, varies significantly with patient characteristics, thus underscoring the need for competition measures to account for both quality differentiation and patient demographics—an aspect our model directly addresses by considering customer- and hospital-specific trade-offs between distance and quality. Subsequently, Gowrisankaran and Town (2003) examined hospital quality decisions in Southern California, observing divergent effects based on patient type: increased competition for HMO patients led to decreased risk-adjusted hospital mortality rates, suggesting improved welfare, while heightened competition for Medicare enrollees was paradoxically associated with *increased* risk-adjusted mortality rates, potentially reducing welfare.

Supporting this mixed picture, Propper, Burgess, and Green (2004) similarly indicated that competition might increase mortality for specific patient groups. Building on this, Propper, Burgess, and Gossage (2008) leveraged a UK policy change to study competition in an environment with limited quality signals where hospitals primarily competed on price. They found a negative relationship between competition and AMI (Acute Myocardial Infarction) mortality, yet observed reduced waiting times, suggesting strategic behavior where hospitals might reduce unmeasured or unobserved quality to improve more easily measured and observed metrics like waiting times. In contrast, Gaynor, Moreno-Serra, and Propper (2013) studied a pro-competitive reform in England (2006) that granted patients greater choice and access to quality information. Their findings revealed that competition, in this context, led to saved lives, shorter lengths of stay, and stable costs, indicating a positive impact on quality without cost inflation. Further support for positive effects comes from Bloom, Propper, Seiler, and Van Reenen (2015), who found that increased competition significantly improved managerial quality (by 0.4 standard deviations) and hospital performance, leading to a 9.7% increase in survival rates for emergency heart attack patients.

The long-term impact of competition was explored by Kessler and McClellan (2000) using data on Medicare beneficiaries with heart attacks, who noted an ambiguous welfare effect in the 1980s that shifted to unambiguously positive in the 1990s, largely attributed to the rising enrollment in managed care organizations (HMOs) during that period. Kessler and Geppert (2005) investigated the effects of hospital competition on care quality and expenditures for elderly heart attack patients, finding that in competitive markets, low-valuation (less severely ill) patients received less intensive treatment with similar outcomes. In contrast, high-valuation patients received more intensive treatment and experienced significantly better health outcomes, thus highlighting variations in quality outcomes based on patient characteristics.

Finally, a broader assessment by Mutter, Wong, and Goldfarb (2008), employing twelve different hospital competition measures, concluded that competition had a positive impact on some quality measures but a negative one on others, reinforcing the complex and often mixed empirical picture. In summary, the empirical evidence on healthcare competition reveals a consistently ambiguous impact on quality. Hospitals appear to engage in strategic behavior, enhancing quality in some dimensions while potentially reducing it in others, reflecting their use of all available quality-related strategic variables to navigate competitive pressures.

Voucher programs aiming to stimulate competition and target subsidies have been evaluated mainly in developing countries and yield heterogeneous results, though some U.S. experiences with targeted subsidies and competitive contracting offer related lessons. Randomized and quasi-experimental evaluations from Bangladesh, Kenya, and Nicaragua demonstrate that vouchers can increase utilization of maternal and reproductive health services and shift patients toward accredited private or non-governmental providers, with improvements in uptake and short-term behaviors (Lim, Dandona, Hoisington, James, Hogan, and Gakidou (2010); Hatt, Makinen, Madhavan, Phillips, Islam, and Islam (2010)). U.S. evidence on competition and contracting suggests that payment design, monitoring, and accreditation critically determine whether competition improves quality or exacerbates inequities (Dafny (2010); Gaynor and Town (2012)). Overall, the literature suggests that competition and vouchers can yield benefits when embedded in robust regulatory, accreditation, and monitoring frameworks; however, they risk adverse consequences for equity and quality where oversight and provider capacity are weak.

# 8 Conclusions

Many policymakers and economists around the world contend that addressing citizens' demands for better-quality services in health, education, transportation, security, and other areas requires introducing private providers and vouchers into markets that have traditionally been served solely by public providers. The empirical evidence in health and education markets is, at best, mixed. Quality sometimes increases and sometimes decreases, depending on the context. The evidence suggests that providers' strategic behavior and customers' preferences are essential drivers of the mixed results, along with institutional settings.

The theoretical analysis presented in this paper carries several important implications for policymakers considering the implementation of voucher programs. Our findings caution against the assumption that introducing private providers and vouchers will automatically improve service quality. The complex interplay of horizontal and vertical differentiation, alongside firm heterogeneity, can lead to non-monotonic relationships between voucher values, prices, and ultimately, service quality. A uniform voucher policy, without careful consideration of the existing market structure, may not only fail to enhance quality but also exacerbate existing inequalities or even reduce quality in certain market segments. Therefore, policymakers must conduct comprehensive market analyses before introducing vouchers, assessing the degree of consumer heterogeneity, the strength of horizontal preference dispersion, monitoring capacity, and the likely strategic responses of both incumbent public and entering private providers to avoid unintended and potentially adverse outcomes." Thus, after all, Friedman's (1955) proposal might not be the tide that raises all boats.

The analysis suggests the following avenues for future research. Firstly, the study should consider the impact of vouchers on market coverage. This will complicate the analysis, as vouchers and quality will affect the extensive margin, not only the intensive margin. If sub-game perfect equilibrium prices are decreasing in the voucher, then the extensive margin will be positive. Yet, it is hard to show that this is the case. Secondly, it would be beneficial to study targeted vouchers by income level. This will prevent subsidizing relatively high-income customers who would patronize an expensive provider without a voucher. This also complicates the analysis by introducing another asymmetric margin

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# A Proofs of Results in Section 4

*Proof of Proposition 1.* Because $G^k$ are identically distributed, for all $j \in \mathcal{J}$ and $j \in \mathcal{J}^0$

$$D^j(p,q) = \mathbb{E}_{\theta,\epsilon}\Big[\prod_{k \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}} G\Big(\triangle U_{jk}(\theta) + \epsilon^j\Big)\Big]$$

and for all for all $j \in \mathcal{J}$,

$\prod_{k \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}} G\Big(\triangle U_{jk}(\theta)\Big)\Big] < 0$

where $\nu_g(\cdot) \equiv g(\cdot)/G(\cdot)$,

$$D^j_k(p,q) = \mathbb{E}_{\theta,\epsilon}\Big[\nu_g\Big(\triangle U_{jk}(\theta)\Big) \times \prod_{k \in \mathcal{J} \cup \mathcal{J}^0 \setminus \{j\}} G\Big(\triangle U_{jk}(\theta) + \epsilon^j\Big)\Big] > 0$$

$D^j(p,q)$ is strictly decreasing in $p^j$ and is strictly increasing in $p_{j'}$.

A log supermodular or MTP (multivariate totally positive of order 2) function is similarly defined for positive functions by $f(x \vee y) f(x \wedge y) > f(x) f(y)$. Thus. $D^j(p,q)$ is log-supermodular if $D^j(p \vee p', q) D^j(p \wedge p', q) > D^j(p,q) D^j(p',q)$. Because the multiplication of $TP_2$ functions is $TP_2$, the demand function is $TP_2$ if $G\Big(\triangle U_{jk}(\theta) + \epsilon^j\Big)$ is $TP_2$ in $(p^j, p_k)$. Observe that $G\Big(\triangle U_{jk}(\theta) + \epsilon^j\Big)$ can be written as $G\Big(K - (p^j - p^k)\Big)$. Let $p^{j'} \geq p^j$ and $p'_k \geq p_k$, this is $TP_2$ if and only if

$$\begin{aligned} & G\Big(K - (p^{j'} - p^{k'})\Big) G\Big(K - (p^j - p^k)\Big) \geq G\Big(K - (p^{j'} - p^k)\Big) G\Big(K - (p^j - p^{k'})\Big) \\ \Longleftrightarrow & G\Big(K - (p^{j'} - p_{k'})\Big) G\Big(K - (p^j - p^k)\Big) \geq G\Big(K - (p^{j'} - p_{k'}) - z^k\Big) G\Big(K - (p^j - p^k) + z^k\Big), \end{aligned}$$

where $z^k = p^{k'} - p^k$. Observe that the RHS is decreasing in $z$ whenever

$$
\begin{aligned}
&-g\Big(K-(p^{j'}-p^{k'})-z_k\Big)G\Big(K-(p^j-p^k)+z_k\Big)+\\
&G\Big(K-(p^{j'}-p^{k'})-z^k\Big)g\Big(K-(p^j-p^k)+z^k\Big)\leq 0\\
&\iff\\
&\nu_g\Big(K-(p^{j'}-p^{k'})+z^k\Big)\leq \nu_g\Big(K-(p^j-p^k)-z^k\Big).
\end{aligned}
$$

Because $G$ is log-concave and $z^k \geq 0$, this holds for all $z^k$. Because the inequality holds with equality when $z^k = 0$, the inequality holds for all $z^k > 0$. Thus, $D^j(p,q)$ is $TP_2$ in $p$. Because TP-2 is preserved under marginalization the demand is log-supermodular in $p$. We can proceed in the same way to show that is log-submodular in $q$.

Let $m = n + N$ and $d^m \in \Re^m$ be a vector of 1s and $b > 0$. Because demand depends on the difference in prices, we have that $D^j(p,q) = D^j(p + bd^m, q)$. Log-concavity in $p^j$ follows from this and the fact that $D^j(p,q)$ decreasing in $p^j$ and is $TP_2$. There is a well-known duality that a positive Lebesgue-measurable function, $f(x)$ on $\Re$, is log concave if and only if $f(x-y)$ is $TP_2$ in $x$ and $y$. Since monotone functions and continuous functions are Lebesgue-measurable, this duality holds for these functions.

Because $D^j(p,q)$ has increasing differences between $p^j$ and $p^{-j}$, for any $p_{jH} > p_{jL}$ and $b_H > b_L$, $D^j(p_{jH}, p^{-j} + b_H d^m, q)D^j(p_{jL}, p^{-j} + b_L d^m, q) \leq D^j(p_{jH}, p^{-j} + b_L d^m, q)D^j(p_{jL}, p^{-j} + b_H d^m, q)$. Because $D^j(p,q) = D^j(p + bd^m, q)$, we get that $D^j(p_{jH} - b_H, p^{-j}, q)D^j(p_{jL} - b_L, p^{-j}, q) \geq D^j(p_{jH} - b_L, p^{-j}, q)D^j(p_{jL} - b_H, p^{-j}, q)$. Hence, this implies that $D^j(p,q)$ is $TP_2$ in $p^j$ and $b$. Since $D^j(p,q)$ is decreasing in $p^j$, it is Lebesgue-measurable and therefore by Hardy, Littlewood, Pólya, Pólya, et al.'s (1952) result, $D^j(p,q)$ is log-concave by the duality between log concave functions and $TP_2$ functions.

Log-supermodularity in $(p^j, q^j)$ and log-submodularity readily follows from the log-concavity of $D^j(p,q)$ in $p$ and $q$.

$$
D^j_{q^j}(q)\big|_{p^j=k} = \mathbb{E}_{\theta,\epsilon}\left[\theta \sum_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} \nu_g\Big(\triangle U_{jk}(\theta)+\epsilon^j\Big) \times \prod_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} G\Big(\triangle U_{jk}(\theta)+\epsilon^j\Big)\right] > 0,
$$

for $k \in \mathcal{J}$

$$D^j_{q^k}(q)\big|_{p^j=k} = -\mathbb{E}_{\theta,\epsilon}\Big[\theta \nu_g\Big(\triangle U_{jk}(\theta) + \epsilon^j\Big) \times \prod_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} G\Big(\triangle U_{jk}(\theta) + \epsilon^j\Big)\Big] < 0$$

Proceeding as before we can show that $D^j(p,q)$ is TP2 in $(p^j, q^j)$ and in $(p^j, -q^{-j})$.

Observe that for all $j, j' \in \mathcal{J} \cup \mathcal{J}^0$, is log-supermodular in $p$ if for each pair $p^j, p_{j'}$, the following holds

$$\frac{1}{D^j(p,q)} D^j_{j,j'}(p,q) - \frac{1}{(D^j(p,q))^2} D^j_j(p,q) D^j_{j'}(p,q) \geq 0.$$

Observe that

$$D^j_{jj}(p,q) = \mathbb{E}_{\theta,\epsilon}\Bigg[\Bigg(\sum_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} \nu'_g\Big(\triangle U_{jk}(\theta) + \epsilon^j\Big) + \Big(\sum_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} \nu_g\Big(\triangle U_{jk}(\theta) + \epsilon^j\Big)\Big)^2\Bigg) \times \\ \prod_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} G\big(\triangle U_{jk}(\theta) + \epsilon^j\big)\Bigg],$$

$$D^j_{jk}(p,q) = -\mathbb{E}_{\theta,\epsilon}\Bigg[\Bigg(\nu'_g\big(\triangle U_{jk}(\theta) + \epsilon^j\big) + \nu_g\big(\triangle U_{jk}(\theta) + \epsilon^j\big) \times \sum_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} \nu_g\Big(U\triangle U_{jk}(\theta)\Big)\Bigg) \times \\ \prod_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} G\Big(\triangle U_{jk}(\theta) + \epsilon^j\Big)\Bigg],$$

where $\nu'_g < 0$ because $g(\cdot)$ is log-concave.

□

*Proof of Proposition 2.* The proof follows closely the proof of Theorem 1 in Mizuno (2003).

**Lemma 2.** *If $D(p,q)$ is strictly decreasing and log-concave in $p$, then for each $p^{-j}$,*

$$\Pi(p,q) \equiv (p^j + v - c(q^j))D(p,q),$$

*is continuous and strictly quasi-concave, and there is a unique $p$ that maximizes $\Pi(p,q)$.*

*Proof of Lemma 2.* In the proof of this lemma, we suppress $p^{-j}$ and the price subscripts and firms' superscripts for simplicity. Continuity is immediate, since convex functions and log-concave functions are continuous.

If $D(p,q)$ is strictly positive and log-concave in $p$, $D(p,q)$ is strictly convex in $p$ because

$$D(\lambda p_1 + (1-\lambda)p_2, q) > D(p_1,q)^{\lambda} D(p_2,q)^{1-\lambda},$$

for $D(p_1,q) \neq D(p_2,q)$ and $\lambda \in (0,1)$.

Since $D(p,q)$ is strictly decreasing and convex, it has a strictly increasing and concave inverse $k(x)$, where $pD(p) = \frac{k(x)}{x}$. Let $z = \frac{1}{x} = \frac{1}{D(p,q)}$, then

$$pD(p,q) = \frac{c(x)}{x} = zc\left(\frac{1}{z}\right),$$

so that $k\left(\frac{1}{z}\right)$ is the inverse demand function. Since $k(x)$ is strictly increasing and strictly concave, $(x+v)k\left(\frac{1}{x}\right)$ is strictly concave. Hence

$$\Pi(p,q) = pD(p,q) + (v - c(q))D(p,q) = (z + v - c(q))k\left(\frac{1}{z}\right)$$

is strictly concave as the sum of strictly concave functions is strictly concave.

Since $D(p,q)$ is strictly decreasing, if $\Pi$ is strictly concave in demand, it is strictly quasi-concave in price. Hence, a maximizer is unique if it exists.

Since $D(p,q)$ is strictly decreasing and log-concave,

$$\lim_{p\to\infty} D(p,q) = \lim_{p\to\infty} pD(p,q) = 0, \quad \text{so that} \quad \lim_{p\to\infty} \Pi(p,q) = 0.$$

Because $D(p,q)$ is strictly positive and strictly decreasing, $\Pi(\hat{p},q) > 0$ for a sufficiently large $p$ such that $p + v - c(q) >> 0$, so that we can take a $\hat{p}$ such that $\Pi(\hat{p},q) > \epsilon$.

Since $\lim_{p\to\infty} \Pi(p,q) = (p + v - c(q^j))D(p,q) = 0$, there is a $\bar{p}$ such that

$$\Pi(\hat{p},q) > \Pi(p,q)$$

for all $p > \bar{p}$.

Since $\Pi(0, q) > \Pi(p, q)$ for any $0 > p$, any maximizer of $\Pi(p, q)$ is in $[0, \bar{p}]$, and it exists since $\Pi(p, q)$ is continuous and $[0, \bar{p}]$ is compact. We can proceed exactly in the same way with $\Pi(p, q; \beta^j)$

$\square$

Let $R^j(p)$ be provider $j$'s best response.

**Lemma 3.** *Suppose that $D(p^j, p^{-j}, q)$ is strictly positive, strictly decreasing in $p$, $C(\cdot)$ is increasing and convex, and $\ln D(p^j, p^{-j}, q)$ has increasing differences. Then if $D(p^j, p^{-j}, q)$ is increasing in $p$ or $C(\cdot)$ is linear, $R(p)$ is increasing.*

*Proof of Lemma 3.* Since any function on the real line is quasi-supermodular, it is sufficient to show that $D(p^j, p^{-j})$ has the single-crossing property in order to apply Milgrom and Shannon's monotonicity theorem. If $D(p^j, p^{-j})$ does not have the single-crossing property, there exist $p^{-jH} \geq p^{-jL}$ and $p^{jH} > p^{jL}$ such that

$$(p^{jL} + v - c(q^j))D(p^{jL}, p^{-jH}, q) > (p^{jH} + v - c(q^j))D(p^{jH}, p^{-jH}, q) \tag{A.1}$$

and

$$(p^{jH} + v - c(q^j))D(p^{jH}, p^{-jL}, q) > (p^{jL} + v - c(q^j))D(p^{jL}, p^{-jL}, q) \tag{A.2}$$

By multiplying (A.2) by $D(p^{jL}, p^{-jL}, q) - D(p^{jH}, p^{-jL}, q) > 0$ and (A.2) by $D(p^{jL}, p^{-jH}, q) - D(p^{jH}, p^{-jH}), q) > 0$, and adding up, we obtain:

$$(p^{jH} - p^{jL})(D(p^{jL}, p^{-jH}, q)D(p^{jH}, p^{-jL}, q) - D(p^{jL}, p^{-jL}, q)D(p^{jH}, p^{-jH}, q)) > 0$$

The left-hand side is non-positive because $p^H > p^L$ and the log-supermodularity of demand implies that $D(p^{jH}, p^{-jH}, q)D(p^{jL}, p^{-jL}, q) \geq D(p^{jL}, p^{-jH}, q)D(p^{jH}, p^{-jL}, q)$. This contradicts the hypothesis that profits do not satisfy the single-crossing property.

$\square$

We can replicate these proofs for public providers and show their best responses are increasing functions as for private providers.

Let $R^j(p^{-j})$ be private provider $j$'s best response function (it is unique) and $R^{j0}(p^{-j})$ be public provider $j$'s best response function (it is unique). Let $d^m$, with $m = n + N$, be a vector of 1s and $b > 0$. Because demands depend on the difference in prices, we have that $D^j(p) = D^j(p + bd^m)$. Then, for all $\beta^j \in [0, 1]$,

$$\begin{aligned}
&\big(\beta^j(R^j(p^{-j}) + v - c_j) + 1 - \beta^j\big)D^j(R^j(p^{-j}), p^{-j})\\
>&\big(\beta^j((R^j(p^{-j} + bd^{m-1}) - b) + v - c_j)) + 1 - \beta^j\big)D^j(R^j(p^{-j} + bd^{m-1}) - b, p^{-j}), \text{ bc max}\\
=&\big(\beta^j((R^j(p^{-j} + bd^{m-1}) - b) + v - c_j) + 1 - \beta^j\big)D^j(R^j(p^{-j} + bd^{m-1}), p^{-j} + bd^{m-1}),\\
&\text{bc } D^j(p) = D^j(p + bd^m)
\end{aligned}$$

and

$$\begin{aligned}
&\big(\beta^j(R^j(p^{-j} + bd^{m-1}) + v - c_j) + 1 - \beta^j\big)D^j(R^j(p^{-j} + bd^{m-1}), p^{-j} + bd^{m-1})\\
>&\big(\beta^j(R^j(p^{-j}) + b) + v - c_j) + 1 - \beta^j\big)D^j(R^j(p^{-j} + bd^{m-1}), p^{-j} + bd^{m-1}), \text{ bc max}\\
=&\big(\beta^j(R^j(p^{-j}) + b) + v - c_j) + 1 - \beta^j\big)D^j(R^j(p^{-j}), p^{-j}), \text{ bc } D^j(p) = D^j(p + bd^m)
\end{aligned}$$

We deduce from these two inequalities that

$$\begin{aligned}
0 >&b(D^j(R^j(p^{-j} + bd^{m-1}), p^{-j} + bd^{m-1}) - D^j(R^j(p^{-j}), p^{-j}))\\
=&b(D^j(R^j(p^{-j} + bd^{m-1}) - b, p^{-j}) - D^j(R^j(p^{-j}), p^{-j})).
\end{aligned}$$

Because demand is strictly decreasing, this implies that the best-response is single valued for each $j$ and $R^j(p^{-j} + bd^{m-1}) < R^j(p^{-j}) + b$ for all $b > 0$.

Next, let's assume that there are two fixed points, denoted by $x$ and $y$. Let $e \equiv \max_{j\in\{1,\dots,m\}} |x^j - y^j|$. Hence, $R(p)$ has two different fixed points. Observe that $x^{-j} \wedge y^{-j} \leq y^{-j}$, $x^{-j} \vee y^{-j} \geq y^{-j}$ and $x^{-j} \vee y^{-j} \leq x^{-j} \wedge y^{-j} + ed^{m-1}$. Because $R^j(x^{-j} + bd^{m-1}) < R^j(x^{-j}) + e$ and $R^j$ is increasing $|R^j(x^{-j}) - R^j(y^{-j})| \leq (R^j(x^{-j} \wedge y^{-j} + bd^{m-1}) - R^j(x^{-j} \wedge y^{-j})) < e$. This contradicts the fact that $|R^j(x) - R^j(y)| = |x - y| = e$. □

*Proof of Proposition 4.* Take any matrix $A$ which entries $a_{ij}$. Then matrix $A$ is $B_0$-matrix if and only if for all $i \in \mathcal{J}$, we have that

$$\sum_{j=1}^{n} a_{ij} \geq n \max \left\{0, a_{ij} | j \neq i\right\}.$$

Let $b$ be a matrix with elements $b_{ij}$ and matrix $A(b)_{jk}$ be the matrix resulting from substituting column $j$ per vector $b^k$. Then Christensen (2018) shows that $(A(b)_{jk})^T$ is a B-matrix if vector $b$ satisfies the following

$$\sum_{i=1}^{n} b_{ik} \geq n \max \left\{0, b_{ik} | i \neq k\right\},$$

Let $y$ be a matrix with elements $y_{ij}$. Thus, if we have the system of equations $Ay = b$, using the Cramer's rule, we can show that

$$y_{jk} = \frac{\det\left(A(b)_{jk}\right)}{\det\left(A\right)},$$

and therefore $y_{jk} \geq 0$ if $A$ and $A(b)_{jk}$ are $B_0$-matrices. Theorem 1 in Christensen (2018) shows that $y_i > 0$ whenever $A^T$ is a $B_0$ matrix and $b$ is mean positive in $i$.

It is easy to check that $-H$ is a $B_0$-matrix since $H$ is diagonally dominant and in each row, the off-diagonal elements of $-H$ are all negative and smaller than the diagonal element. Similarly, we can check that $(-H)^T$ is a B-matrix. Let $-H^{i,j}$ the co-factor $ij$ from matrix $-H$. Hence, $\sum_{j=1}^{0}(-H^{i,j}) \geq 0$ and $\det\left(-H\right) > 0$.

To see that $-\Pi$ is a B-matrix, observe that this requires that for all $i \in \mathcal{J}$,

$$-\sum_{j \in \mathcal{J} \cup \mathcal{J}^0} \frac{\partial^2 \log \Pi_i(p, q; \beta^j)}{\partial p_i \partial p^j} \geq n \max_{j \in \mathcal{J} \cup \mathcal{J}^0 \setminus i} \left\{0, -\frac{\partial^2 \log \Pi_i(p, q)}{\partial p_i \partial p^j} | j \neq i\right\},$$

which follows from the fact that

$$-\frac{\partial^2 \log \Pi_i(p, q; \beta^j)}{\partial p_i^2} > \sum_{j \in \mathcal{J} \cup \mathcal{J}^0 \setminus i} \frac{\partial \log \Pi_i(p, q)}{\partial p_i \partial p^j},$$

and

$$-\frac{\partial^2 \log \Pi_i(p,q;\beta^j)}{\partial p_i \partial p^j} < 0, \; \forall j \neq i.$$

To see that $-H^T$ is a B-matrix, observe that this requires that for all $i \in \mathcal{J}$,

$$-\sum_{j \in \mathcal{J} \cup \mathcal{J}^0} \frac{\partial^2 \log \Pi^j(p,q;\beta^j)}{\partial p^j \partial p_i} \geq n \max_{j \in \mathcal{J} \cup \mathcal{J}^0 \setminus i} \left\{0, -\frac{\partial^2 \log \Pi^j(p,q;\beta^j)}{\partial p^j \partial p_i} | j \neq i\right\},$$

which follows from the fact that

$$-\frac{\partial^2 \log \Pi_i(p,q;\beta^j)}{\partial p_i^2} > \sum_{j \in \mathcal{J} \cup \mathcal{J}^0 \setminus i} \frac{\partial \log \Pi^j(p,q;\beta^j)}{\partial p^j \partial p_i},$$

and

$$-\frac{\partial^2 \log \Pi^j(p,q;\beta^j)}{\partial p^j \partial p_i} < 0, \; \forall j \neq i.$$

Let's also define the matrix $-H(p,q;\beta^j)$ as the matrix with entries $\left\{\frac{\partial^2 \log \Pi_i(p,q;\beta^j)}{\partial p_i \partial q^j}\right\}_{i,j \in \mathcal{J}}$. Using Cramer's rule, we can show that

$$\frac{\partial p^j(q)}{\partial q^k} = \frac{\det\left(-H^{jk}(p,q;\beta)\right)}{\det\left(-H(p,q;\beta)\right)},$$

where $-H^{jk}(p,q;\beta)$ is the matrix obtained from $-H$ by replacing column $j$ with the column vector $k$ from $H(p,q;\beta)$.

Then, $-H^{jk}(p,q;\beta)$ is a B-matrix if and only if

$$\sum_{j \in \mathcal{J} \cup \mathcal{J}^0} \frac{\partial^2 \log \Pi^j(p,q;\beta^j)}{\partial p^j \partial q^k} \geq n \max\left\{0, \frac{\partial^2 \log \Pi^j(p,q;\beta^j)}{\partial p^j \partial q^k} | j \neq k\right\}, \tag{A.3}$$

If this holds $\det -H(p,q;\beta) > 0$.

Let's assume that $m = n + N$ and define that $\Pi^j_{xy} \equiv \partial^2 \log \Pi^j / \partial x \partial y$, then from the equilibrium conditions we deduce the following

$$H_{m,m}(p) \equiv \begin{pmatrix} \Pi^1_{1,1} & \Pi^1_{1,2} & \dots & \dots & \dots & \dots & \Pi^1_{1,m} \\ \Pi^2_{2,1} & \Pi^2_{2,2} & \dots & \dots & \dots & \dots & \Pi^2_{2,m} \\ \vdots & \dots & \ddots & \dots & \dots & \dots & \vdots \\ \Pi^k_{k,1} & \ddots & \dots & \Pi^k_{k,k} & \dots & \dots & \Pi^k_{k,m} \\ \\ \Pi^{k+1}_{k+1,1} & \ddots & \dots & \dots & \Pi^{k+1}_{k+1,k+1} & \dots & \Pi^{k+1}_{k+1,m} \\ \vdots & \dots & \dots & \dots & \dots & \ddots & \dots \\ \Pi^m_{m,1} & \dots & \dots & \dots & \dots & \dots & \Pi^m_{m,m} \end{pmatrix}$$

$$p_{q^j} \equiv \begin{pmatrix} p^1_{q^j} \\ \vdots \\ \vdots \\ p^n_{q^j} \\ \vdots \\ \vdots \\ p^m_{q^j} \end{pmatrix} \qquad b(q^j) \equiv \begin{pmatrix} -\Pi^1_{1,q^j} \\ \vdots \\ \vdots \\ -\Pi^n_{n,q^j} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$H_{m,m}(p) p_{q^j} = b(q^j) \tag{A.4}$$

It follows then

$$p^i_{q^j} = -\frac{\sum_{i \in \mathcal{J} \cup \mathcal{J}^0} \Pi^i_{i,q^j} H^{ij}(p,q;\beta)}{\sum_{i \in \mathcal{J} \cup \mathcal{J}^0} \Pi^i_{i,j} H^{ij}(p,q;\beta)},$$

where $H^{ij}(p,q;\beta)$ is the co-factor $ij$.

Hence, this is non-negative if and only if $-\sum_{i \in \mathcal{J} \cup \mathcal{J}^0} \Pi^i_{i,j} H^{ij}(p,q;\beta) \geq 0$, which is the case when $H(p,q;\beta)$ is a $B_0$-matrix. This follows from substituting the $i$ row by a row of $-1$ and then the determinant of this matrix is $-\sum_{i \in \mathcal{J} \cup \mathcal{J}^0} H^{ij}(p,q;\beta)$, which is positive because the new matrix with -1s in row 1 is a $B_0$ matrix since the row sum of -1s is lower than or equal to $n$ times the lowest between zero and the smallest off-diagonal row element, which is $-1$.

If the Jacobian of the equilibrium conditions and its transpose are both $B_0$-matrices as it is the case here, then its sum is negative definite (see, Christensen (2018)) and, thereby, Rosen's (1965) diagonally strict concavity property holds. Hence, the equilibrium will be unique.

$\square$

*Proof of Corollary 4.* Let's define $\triangle U(p,q;\theta) \equiv U(y,q^1,p^1,\theta) - U(y,q^0,p^0,\theta)$.

Let's define

$$M^1(p^1,p^0,q) \equiv \frac{1}{p^1+v-c(q^1)} + \frac{\mathbb{E}_{\theta,\epsilon}\Big[G^{n-1}(\epsilon)G^N\big(\triangle U(p,q;\theta)+\epsilon\big)\Big((n-1)\nu_g(\epsilon)+N\nu_g\big(\triangle U(p,q;\theta)+\epsilon\big)\Big)\Big]}{\mathbb{E}_{\theta,\epsilon}G^{n-1}(\epsilon)G^N\big(\triangle U(p,q;\theta)+\epsilon\big)}$$

and

$$M^0(p^1,p^0,q) \equiv \frac{\beta}{\beta(p^0+g-c(q^0))+1-\beta} + \frac{\mathbb{E}_{\theta,\epsilon}\Big[G^{N-1}(\epsilon)G^n\Big(-\triangle U(p,q;\theta)+\epsilon\Big)\Big((N-1)\nu_g(\epsilon)+n\nu_g\big(-\triangle U(p,q;\theta)+\epsilon\big)\Big)\Big]}{\beta\mathbb{E}_{\theta,\epsilon}G^{N-1}(\epsilon)G^n\big(-\triangle U(p,q;\theta)+\epsilon\big)}.$$

In this case, the equilibrium price profile is the unique solution to the following system of equations $M^1(p^1,p^0,q)=0$ and $M^0(p^1,p^0,q)=0$.

Observe that if $n=N$ and $g=v-\frac{1-\beta}{\beta}$, then at $q^1=q^0=\hat{q}$, $M^1(p^1(\hat{q},\hat{q}),p^0(\hat{q},\hat{q}),\hat{q},\hat{q}) = M^0(p^1(\hat{q},\hat{q}),p^0(\hat{q},\hat{q}),\hat{q},\hat{q})=0$.

Let's consider $q^1 > q^0 = \hat{q}$ and assume that the new prices are: $p^1(q^1,\hat{q}) < p^1(\hat{q},\hat{q})$

and $p^0(q^1,\hat{q}) > p^0(\hat{q},\hat{q})$.

$$\begin{aligned}
0 =& M^1(p^1(\hat{q},\hat{q}), p^0(\hat{q},\hat{q}), \hat{q}, \hat{q}) \\
\leq& M^1(p^1(q^1,\hat{q}), p^0(\hat{q},\hat{q}), \hat{q}, \hat{q}) \text{ by } \log-\text{concavity in } \mathrm{p}^1 \\
\leq& M^1(p^1(q^1,\hat{q}), p^0(\hat{q},\hat{q}), q^1, \hat{q}) \text{ by } \log-\text{supermodularity in } (\mathrm{p}^1, \mathrm{q}^1) \\
\leq& M^1(p^1(q^1,\hat{q}), p^0(q^1,\hat{q}), \hat{q}, \hat{q}) \text{ by } \log-\text{supermodularity in } (\mathrm{p}^1, \mathrm{p}^0)
\end{aligned}$$

$$\begin{aligned}
0 =& M^0(p^1(\hat{q},\hat{q}), p^0(\hat{q},\hat{q}), \hat{q}, \hat{q}) \\
\geq& M^0(p^1(\hat{q},\hat{q}), p^0(q^1,\hat{q}), \hat{q}, \hat{q}) \text{ by } \log-\text{concavity in } \mathrm{p}^1 \\
\geq& M^0(p^1(\hat{q},\hat{q}), p^0(q^1,\hat{q}), q^1, \hat{q}) \text{ by } \log-\text{submodularity in } (\mathrm{p}^0, \mathrm{q}^1) \\
\geq& M^0(p^1(q^1,\hat{q}), p^0(q^1,\hat{q}), \hat{q}, \hat{q}) \text{ by } \log-\text{supermodularity in } (\mathrm{p}^1, \mathrm{p}^0)
\end{aligned}$$

We deduce from this that $q^1 > q^0 = \hat{q}$, $p^1(q^1,\hat{q}) < p^1(\hat{q},\hat{q})$, and $p^0(q^1,\hat{q}) > p^0(\hat{q},\hat{q})$ cannot be an equilibrium.

Let $\Pi^j_{xy} = \partial^2 \log \Pi^j / \partial x \partial y$. Totally differentiating the FOC for private and public firms and imposing symmetry, we obtain that

$$p^1_v(q) = -\frac{\Pi^1_{1v}(\Pi^0_{00} + (N-1)\Pi^0_{0N})}{(\Pi^0_{00} + (N-1)\Pi^0_{0N})(\Pi^1_{11} + (n-1)\Pi^1_{1n}) - nN\Pi^0_{01}\Pi^1_{10}} < 0,$$

where the sign follows because $\Pi^1_{1v} < 0$ and the $B_0$-matrix assumption that implies that the numerator is negative and denominator is positive. Also, we obtain that

$$p^0_v(q) = \frac{n\Pi^1_{1v}\Pi^0_{01}}{(\Pi^0_{00} + (N-1)\Pi^0_{0N})(\Pi^1_{11} + (n-1)\Pi^1_{1n}) - nN\Pi^0_{01}\Pi^1_{10}} < 0$$

because $\Pi^0_{01} \geq 0$.

It follows from this that

$$p_v^1(q) - p_v^0(q) = -\frac{\Pi_{1v}^1((\Pi_{00}^0 + (N-1)\Pi_{0N}^0) + n\Pi_{01}^0)}{(\Pi_{00}^0 + (N-1)\Pi_{0N}^0)(\Pi_{11}^1 + (n-1)\Pi_{1n}^1) - nN\Pi_{01}^0\Pi_{10}^1} < 0,$$

since the numerator and the denominator are both positive because of the $B_0$-matrix property.

Totally differentiating the FOC for private and public firms with respect to $q^1$ and imposing symmetry, we obtain that

$$p_{q^1}^1(q) = -\frac{\Pi_{1q^1}^1(\Pi_{00}^0 + (N-1)\Pi_{0N}^0) - N\Pi_{0q^1}^0\Pi_{10}^1}{(\Pi_{00}^0 + (N-1)\Pi_{0N}^0)(\Pi_{11}^1 + (n-1)\Pi_{1n}^1) - nN\Pi_{01}^0\Pi_{10}^1} > 0.$$

where the sign follows because $H_{q^1}$ is mean positive dominant and $H_{q^1}^T$ is a $B_0$-matrix, which implies that the numerator is negative and denominator positive. □

*Proof of Corollary 5.* Also, we obtain that

$$p_{q^1}^0(q) = \frac{-\Pi_{0q^1}^0(\Pi_{00}^0 + (N-1)\Pi_{0N}^0) + n\Pi_{1q^1}^1\Pi_{01}^0}{(\Pi_{00}^0 + (N-1)\Pi_{0N}^0)(\Pi_{11}^1 + (n-1)\Pi_{1n}^1) - nN\Pi_{01}^0\Pi_{10}^1}.$$

Substituting into for $p_{q^1}^1(q)$, this is negative whenever $n\Pi_{1q^1}^1\Pi_{01}^0 \leq \Pi_{0q^1}^0(\Pi_{11}^1 + (n-1)\Pi_{1n}^1)$. If $\Pi_{0q^1}^0 \geq 0$, this never holds, whereas if $\Pi_{0q^1}^0 < 0$, we deduce the result from the inequality.

It follows from this that

$$p_{q^1}^1(q) - p_{q^1}^0(q) =$$
$$-\frac{\Pi_{1q^1}^1(\Pi_{00}^0 + (N-1)\Pi_{0N}^0) - N\Pi_{0q^1}^0\Pi_{10}^1 - \Pi_{0q^1}^0(\Pi_{00}^0 + (N-1)\Pi_{0N}^0) + n\Pi_{1q^1}^1\Pi_{01}^0}{(\Pi_{00}^0 + (N-1)\Pi_{0N}^0)(\Pi_{11}^1 + (n-1)\Pi_{1n}^1) - nN\Pi_{01}^0\Pi_{10}^1} > 0.$$

Because of the $B_0$ property and $H_q$ is mean-positive dominant.

□

*Proof of Lemma 1.* Let's define $m \equiv \min_{\epsilon\in[\underline{\epsilon},\bar{\epsilon}]} g(\epsilon)$ and $M \equiv \max_{\epsilon\in[\underline{\epsilon},\bar{\epsilon}]} |g'(\epsilon)|$. Because $g'$ is bounded on $[\underline{\epsilon}, \bar{\epsilon}]$ and $M < \infty$.

Then, $\prod_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} G\big(\triangle U_{jk}(\theta)+\epsilon^j\big)$ is convex if and only if

$$\sum_{k\neq j}\prod_{k\in\mathcal{J}\cup\mathcal{J}^0\setminus\{j\}} G\big(\triangle U_{jk}(\theta)+\epsilon^j\big)\Big(v_g'(\triangle U_{jk}+\epsilon^j)+v_g(\triangle U_{jk}+\epsilon^j)\sum_{h\neq j}v_g(\triangle U_{jh}+\epsilon^j)\Big)>0.$$

Observe

$$\begin{aligned}
&v_g'(\triangle U_{jk}+\epsilon^j)+v_g(\triangle U_{jk}+\epsilon^j)\sum_{h\neq j}v_g(\triangle U_{jh}+\epsilon^j)\\
=&\frac{1}{G(\triangle U_{jk}+\epsilon^j)}\Big(g'(\triangle U_{jk}+\epsilon^j)+g(\triangle U_{jk}+\epsilon^j)\sum_{h\neq j}v_g^h(\triangle U_{jh}+\epsilon^j)\Big)\\
\geq&\frac{1}{G(\triangle U_{jk}+\epsilon^j)}\Big(-M+m^2\sum_{h\neq j}\frac{1}{G(\triangle U_{jh}+\epsilon^j)}\Big)\\
\geq&\frac{1}{G(\triangle U_{jk}+\epsilon^j)}\Big(-M+(n-1)m^2\Big)
\end{aligned}$$

For $m$ sufficiently large, this is positive. $\square$